PMATH 352, Spring 2011

Assignment #1

Due at 10:30am on Wednesday, May 11th, 2011

Problem 1. Let $V \subset \mathbb{C}$ be open and $z \in \mathbb{C}$.

(a) If $f : V \to \mathbb{C}$ is $\mathbb{C}$-differentiable at z, show that f is continuous at z.

(b) Prove the *product rule*: if $f, g : V \to \mathbb{C}$ are each $\mathbb{C}$-differentiable at z, then so too is fg with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

Problem 2. Let $V \subset \mathbb{C}$ be open and $z \in \mathbb{C}$.

(a) Show that $f : V \to \mathbb{C}$ is differentiable at z if and only if there is $a \in \mathbb{C}$ and a function $E : D(0,r) \to \mathbb{C}$ (where $r > 0$ is such that $D(z,r) \subset V$) such that

$$f(z+h) = f(z) + ah + E(h) \text{ for } h \in D(0,r), \quad E(0) = 0 \text{ and } \lim_{h \to 0} \frac{E(h)}{h} = 0.$$

(b) Prove that for the "error" function E, in (a), above, that for any $\epsilon > 0$ there is $\delta > 0$ such that for $|h| < \delta$ we have $|E(h)| \leq \epsilon |h|$.

(c) Prove the *chain rule*: If $f : V \to \mathbb{C}$ is differentiable at z, U is an open set containing $f(z)$ and $g : U \to \mathbb{C}$ is $\mathbb{C}$-differentiable at $f(z)$, then the composition $g \circ f$ is $\mathbb{C}$-differentiable at z with

$$(g \circ f)'(z) = g'\big(f(z)\big)g'(z).$$

(d) Let $q : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ be given by $q(z) = \frac{1}{z}$. Show that $q'(z)$ exists for each $z \neq 0$ and is equal to $-\frac{1}{z^2}$. Deduce from this, and from the differentiation rules above, the *quotient rule*: if $f, g : V \to \mathbb{C}$ are $\mathbb{C}$-differentiable at $z \in V$ and $g(z) \neq 0$, then f/g is $\mathbb{C}$-differentiable at z with

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}.$$

Problem 3. Define $\phi(z) = i\dfrac{1-z}{1+z}$ for $z \neq -1$.

(a) Let $\lambda(z) = (1+iz)/(1-iz)$. Show that $\lambda(\phi(z)) = z$, and that ϕ maps $\mathbb{C} \setminus \{-1\}$ one-to-one and onto $\mathbb{C} \setminus \{-i\}$.

(b) Show that ϕ maps the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ one-to-one and onto the upper half plane $\mathbb{H} = \{z : \operatorname{im} z > 0\}$. (Hint: consider z in polar forms)

Problem 4. Let $\mathbb{A} = \{z \in \mathbb{C} : \frac{1}{2} < |z| < 2\}$, and let $f(z) = z + \frac{1}{z}$. Show that $f(\mathbb{A})$ is the interior of an ellipse. (Hint: consider the image of circles)

Problem 5.

Find all complex numbers z such that $z^8 + 16z^4 + 256 = 0$. Write your answers in the standard form.