PMATH 352, Spring 2011

Assignment #2

Due at 10:30am on Wednesday, May 25th, 2011

Problem 1. Let Ω be a rectangle or disk and $f : \Omega \to \mathbb{C}$ be a holomorphic function with real values; i.e., $f(z) \in \mathbb{R}$ for all $z \in \Omega$. Prove that f is a constant.

Problem 2. f $V \subset \mathbb{C}$ is open, $f : V \to \mathbb{C}$, $z \in V$ and $\zeta \in \mathbb{T} = \{\zeta \in \mathbb{C}, |\zeta| = 1\}$, let the *directional derivative* be given by

$$D_\zeta f(z) = \lim_{\substack{t \to 0 \\ t \in \mathbb{R}}} \frac{f(z + t\zeta) - f(z)}{t}$$

when the limit exists. Show that if f is holomorphic at z then $D_\zeta f(z) = f'(z)\zeta$ for each $\zeta \in \mathbb{T}$.

Remark: If $f'(z) \neq 0$ then one can interpret for $\zeta, \xi \in \mathbb{T}$, the ratio $D_\zeta f(z)/D_\xi f(z)$, which by (a) is $\zeta\bar{\xi}$, as the angle made by the curves $t \mapsto f(z + t\zeta)$ and $t \mapsto f(z + t\xi)$. Thus it is often said that "f preserves angles at z", or "f is *conformal* at z".

Problem 3. Given a sequence $(c_n)_{n=1}^\infty \subset \mathbb{C}$ define

$$R = \sup\left\{r \geq 0 : (r^n c_n)_{n=1}^\infty \text{ is bounded in } \mathbb{C}\right\}.$$

(a) Prove that if $R > 0$ then

$$\limsup_{n \to \infty} \sqrt[n]{|c_n|} = \begin{cases} 1/R & \text{if } R < \infty \\ 0 & \text{if } R = \infty \end{cases}$$

Also show that, $R = 0$ if and only if the sequence $(\sqrt[n]{|c_n|})_{n=1}^\infty$ is unbounded.

(b) Suppose, morerover, that $L = \lim_{n\to\infty} |c_{n+1}|/|c_n|$ exists. Then

$$L = \lim_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = \begin{cases} 1/R & \text{if } R < \infty \\ 0 & \text{if } R = \infty. \end{cases}$$

[For (b), first show that if $r < 1/L$ ($r > 0$ in the case that $L = 0$) then there is N in $\mathbb{N}$, such that for $n \geq N$, $r^{n+1}|c_{n+1}| < r^n|c_n|$ and deduce that $(r^n c_n)_{n=1}^\infty$ is bounded.]

Problem 4.

(a) Let $f(z) = \sum_{n=0}^\infty c_n(z - z_0)^n, g(z) = \sum_{n=0}^\infty d_n(z - z_0)^n$ be power series about z_0 with respective radii of convergence R, S. Compute a power series expansion for fg about z_0 and show that its radius of convergence is at least $\min\{R, S\}$.

(b) Prove that for $w, z \in \mathbb{C}$ that $\exp(w + z) = \exp(w)\exp(z)$. Use this to deduce the fomulas

$$\cos(w + z) = \cos(w)\cos(z) - \sin(w)\sin(z), \text{ and}$$
$$\sin(w + z) = \cos(w)\sin(z) + \sin(w)\cos(z).$$

Problem 5.

The intention of this exercise is to improve your understanding of complex logarithms. Part of the exercise is to read and digest the information given. This material is basic stuff you will need to know.

Every non-zero complex number $w = x + iy$ has its polar form

$$w = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

where $r = |w| = \sqrt{x^2 + y^2} > 0$ and $\theta \in \mathbb{R}$. The *modulus* r is uniquely determined by the non-zero number w, but the *argument* θ is not. In fact, any integer multiple of 2π added to θ will give an alternate argument for w. There is, however, a unique θ in the interval $[-\pi, \pi)$ such that $w = re^{i\theta}$.

(a) Draw a simple picture to show you understand the preceding statement.

Let us denote this chosen argument of w by $\theta = \arg w$. Thus we get a function on the punctured plane

$$\arg : \mathbb{C} \setminus \{0\} \to [\pi, \pi).$$

(b) Show that arg is not continuous at any point on the negative x-axis. That is, if $w = x + i0$ and $x < 0$, show that arg is discontinuous at w.

So we have the polar form $w = re^{i\theta}$ where $\theta = \arg z$ is the unique argument of z in the interval $[-\pi, \pi)$ and $r = |z| > 0$. Since $r = |z| > 0$, there is a unique real number s such that $r = e^s = e^{|z|}$. In other words, $s = \log |z|$.

Thus every non-zero complex w has a unique representation as

$$w = e^s e^{i\theta} = e^{s+i\theta} = e^{\log |z| + i \arg z}.$$

Thus we learn that if S is the horizontal strip of complex numbers defined by

$$S = \{s + i\theta : s \in \mathbb{R} \text{ and } \theta \in [-\pi, \pi)\},$$

then the exponential function

$$\exp : S \to \mathbb{C} \setminus \{0\} \text{ where } z = s + i\theta \mapsto s^{s+i\theta} = e^z,$$

is a bijection. The inverse function of this bijection deserves to be called a *logarithm*. This inverse is the function

$$\log : \mathbb{C} \setminus \{0\} \to S \text{ where } w \mapsto \log |z| + i \arg z.$$

Since, as we saw in exercise (b), the arg function is discontinuous on the negative x-axis, this log function is also discontinuous on the negative x-axis. To avoid this hazard, let us remove the negative x-axis from the domain of log and restrict its domain to the so called *cut plane*:

$$\Omega = \{x + iy : \text{ either } y \neq 0 \text{ or } x > 0\} = \mathbb{C} \setminus \{x + i0 : x \leq 0\}.$$

The restriction of log to the cut plane is a bijection onto the open horizontal strip $\Sigma = \{s+i\theta : -\pi < \theta < \pi\}$. That is,

$$\log : \Omega \to \Sigma \text{ is a bijection.}$$

The job now is to verify that log is holomorphic on the cut plane Ω. We saw in class that if $z = x + iy$ lies in the right-half plane $H = \{x + iy : x > 0\}$, then a suitable formula for $\log z$ is given by

$$\log z = \frac{1}{2} \log(x^2 + y^2) + i \arctan\left(\frac{y}{x}\right).$$

We were then able to use the continuity of the various partial derivatives along with the Cauchy-Riemann equations to show that log is holomorphic on H. We really want log to be holomorphic on the cut plane Ω.

(c) Develop a suitable formula for $\log z$ when z lies in the half-plane K that lies above the x-axis, and from that show log is holomorphic on K.

(d) Develop a suitable formula for $\log z$ when z lies in the half-plane L that lies below the x-axis, and from that show log is holomorphic on L.

Having seen that log is holomorphic on three open sets H, K, L that cover the cut plane Ω we conclude that log is a holomorphic function on the cut plane Ω. This log is known as the *principal branch* of the logarithm. Other branches of logarithms are possible by simply changing the 2π interval in which θ can be uniquely chosen as the argument of a non-zero number w.

(e) Prove that $(\log z)' = \dfrac{1}{z}$ for every z in the cut-plane Ω.

(f) Show there is no function whose derivative is $\dfrac{1}{z}$ for all $z \neq 0$.

Hint. We shall soon see that if a function f on a region has zero derivative, then f is constant of the region. It follows from this that if two functions g, f on a region have identical derivatives, then g and f differ by a constant. A function h whose derivative on a region is f is called a *primitive* for f. Thus primitives for a given function on a region, if they exist, are unique up to a constant.