PMATH 352, Spring 2011

Assignment #3

Due at 10:30am on Wednesday, June 8th, 2011

Problem 1. Find the radius of convergence for the following power series.

(a) $\sum_{n=2}^{\infty}(n!)z^n$.
(b) $\sum_{n=2}^{\infty} n^3 z^n$.
(c) $\sum_{n=2}^{\infty} \left(\frac{n-2}{n}\right)^n z^n$.
(d) $\sum_{n=2}^{\infty} \frac{n!}{n^n} z^n$.

Problem 2. If a_n denotes the sequence defined recursively by,

$$a_0 = 2,\ a_1 = 0, \quad a_{n+1} = 2a_n + 3a_{n-1},\ \text{for } n \geq 1.$$

Find the radius of the series $\sum_{n=0}^{\infty} a_n z^n$.

Problem 3. Using your knowledge of geometric series, find a power series expansion that converges to

$$\frac{1}{(2z-1)(z+2)}$$

on a disk centred at the origin, and specify the radius of convergence.

Problem 4. Let

$$\mathbb{H} := \{z \in \mathbb{C} | \operatorname{Im} z > 0\}$$

be the upper half plane, and

$$\psi(z) := \frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha,\ \beta,\ \gamma,\ \delta$ are complex numbers, and $\alpha\delta - \beta\gamma \neq 0$.

(a) If $\alpha,\ \beta,\ \gamma,\ \delta$ are real and $\alpha\delta - \beta\gamma > 0$, show that $\psi : \mathbb{H} \to \mathbb{H}$ is one-to-one and onto.
(b) Conversely, if ψ maps $\mathbb{H}$ to $\mathbb{H}$ one-to-one and onto, prove that there exist $\alpha,\ \beta,\ \gamma,\ \delta \in \mathbb{R}$ and $\alpha\delta - \beta\gamma > 0$.

Problem 5.

(a) Show that a Möbius map takes $\mathbb{D} = \{z : |z| < 1\}$ onto itself if and only if it has the form $Tz = e^{i\theta}\frac{z-a}{1-\bar{a}z}$ for some $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$.
(b) Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two disjoint circles in $\mathbb{C}$. Show that there is a Möbius map T so that $T(\mathcal{C}_1)$ and $T(\mathcal{C}_2)$ are concentric; i.e, their centers are the same.