PMATH 352, Spring 2011

Assignment #4

Due at 10:30am on Wednesday, June 22th, 2011

Problem 1. Let $f(z) = 2/(z^2 - 1)$. For each $p \neq 0$, give the explicit power series centred at p that represents $f(z)$, and the radius of convergence. As a consequence, $f(z)$ is analytic on $\mathbb{C}\backslash\{\pm 1\}$.

Problem 2. Evaluate the following integrals along the paths indicated.

(a) $\displaystyle\int_\gamma \frac{\cos z}{z^4}\,dz,\ \gamma(t) = e^{it},\ 0 \leq t \leq 2\pi.$

(b) $\displaystyle\int_\gamma \frac{e^{z^2}}{z^2}\,dz,\ \gamma(t) = e^{it},\ 0 \leq t \leq 2\pi.$

(c) $\displaystyle\int_\gamma \frac{z^2 - 1}{z^2 + 1}\,dz,\ \gamma(t) = 2e^{it},\ 0 \leq t \leq 2\pi.$

Problem 3.

If a holomorphic function vanishes on the boundary of a closed disk in its domain Ω, show it vanishes on the full disk.

Problem 4.

Let $f = u + iv$ be a holomorphic function defined on a region Ω. Take a disk $B_R(p) \subseteq \Omega$. Prove that the function u satisfies the following mean-value property:

$$u(p) = \frac{1}{2\pi}\int_0^{2\pi} u(p + re^{it})\,dt, \text{ whenever } r < R.$$

Use this mean value property to show that either u is constant on the boundary of $B_r(p)$ or there are points z, w on the boundary of $B_r(p)$ such that $u(w) < u(p) < u(z)$. This is telling you that the graph of u as a surface over an open region never has a high spot or a low spot.

You will need to recall that if g, h are continuous real valued functions on some interval $[a, b]$ and if $f \leq g$ on the interval and if $g \neq h$ on the interval then $\int_a^b g < \int_a^b h$.

Problem 5. If f is entire and $|f(z)| \leq M(1 + |z|^n)$ for some constant M and some exponent n, show that f is a polynomial of degree less or equal to n.