PMATH 352, Spring 2011

Assignment #5

Due at 10:30am on Wednesday, July 6th, 2011

Problem 1. Maximize and minimize $|z+2|^2$ over the solid triangle with vertices $0, 2$ and i.

Problem 2. An open subset Ω of $\mathbb{R}^n$ is called *star shape* if there is a point $p \in \Omega$ such that for every point $q \in \Omega$, the line segment $\overline{pq} \in \Omega$. Show that every star shape region is simply connected. In particular, every convex region is simply connected.

Problem 3.

(a) Prove that if a region Ω is simply connected, for every $\omega \notin \Omega$ and any closed curve γ in Ω, $\operatorname{Ind}_\gamma \omega = 0$.

(b) Using Cauchy's Theorem show that every holomorphic function f on a simply connected region Ω has a primitive over all of Ω. (Hint: Choosing a basepoint $z_0 \in \Omega$, define $h(z) := \oint_\gamma f(z)dz$, where γ is any piecewise smooth curve from z_0 to z. Using Cauchy's theorem to show that the definition of $h(z)$ is independent of choice of γ. Then as we show in class, $h'(z) = f(z)$ as required.)

Problem 4. Let Ω be a simply connected open set in $\mathbb{C}$, and suppose that $f(z)$ is analytic on Ω and never vanishes. Show that for any positive integer n there is a holomorphic function $g(z)$ on Ω such that $g(z)^n = f(z)$.

Problem 5. How many zeros does $p(z) = z^8 + 2z^4 - 8z^2 + 1$ have in the annulus $\mathbb{A} = \{z \in \mathbb{C} | 1 < |z| < 2\}$? Justify your answer.

Problem 6. If f is entire and $\lim_{|z|\to\infty} \dfrac{f(z)}{z} = 0$, show that f is constant.

Problem 7. Let $p(z) = c\prod_{i=1}^{k}(z-a_i)^{n_i}$ be a non-constant polynomial.

(a) Obtain a simplified formula for $\frac{p'(z)}{p(z)}$.

(b) Suppose that $p'(b) = 0$. Use (a) to express b as a *convex combination* of the zeros $a_1, \ldots, a_k$ of p. i.e. $Z(p') \subseteq \overline{\operatorname{conv}(Z(p))}$, where $Z(f)$ means the zero set of a function f and $\operatorname{conv}(A)$ means the smallest convex set containing A. (Hint: use the formula for $0 = \frac{p'(b)}{p(b)}$ from (a), rationalize each term so that the denominator is positive, and then solve for b.)