PMATH 352, Spring 2011

Assignment #6

Due at 10:30am on Wednesday, July 20th, 2011

Problem 1. Find

(a) $\displaystyle\int_0^\infty \frac{x^2}{(x^2+1)^2}\,dx$

(b) $\displaystyle\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$

(c) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

(d) $\displaystyle\int_0^{\pi/2} \frac{dx}{a+\sin^2 x}$, $\forall a > 0$.

Problem 2. If f is a non-constant entire function, prove that $f(\mathbb{C})$ is dense in $\mathbb{C}$. (Hint. If f is a polynomial, the fundamental theorem of algebra ought to do it. If f is not a polynomial, then the classification of singularities applied to $f(\frac{1}{z})$ should help.)

Problem 3. Show that $\sum_{n=0}^{\infty} \dfrac{1}{n^2+a^2} = \dfrac{\pi}{2a}\coth \pi a + \frac{1}{2a^2}$ for $a > 0, a \notin \mathbb{Z}$.

Problem 4. If n is a positive integer, show that $\int_0^\infty \dfrac{1}{1+x^n}\,dx = \dfrac{\pi/n}{\sin(\pi/n)}$. (Hint: Try the pie shaped path from 0 to r to $re^{2\pi i/n}$ and back to 0, then let $r \to \infty$.)

Problem 5.

(a) Let $f(z) = \dfrac{1}{z(z^2-1)}$ for $z \in \mathbb{C}\backslash\{0, 1, -1\}$. Compute the Laurent series for f on each of the annuli $\mathbb{A}_1 = \{z \in \mathbb{C} | 0 < |z-1| < 1\}$, $\mathbb{A}_2 = \{z \in \mathbb{C} | 1 < |z-1| < 2\}$, $\mathbb{A}_3 = \{z \in \mathbb{C} | 0 < |z| < 1\}$,

(b) Let $\gamma(t) = 1/2 + \cos(4t) + 3\sin(4t)i$, $t \in [0, 2\pi]$. Compute $\mathrm{Ind}(\gamma, z)$ where $z \in \{-1, 0, 1\}$.

(c) Compute $\displaystyle\int_\gamma \frac{dz}{z(z^2-1)}$.

Problem 6.

Classify the singularities, including orders of poles, for these functions:

(a) $\cot z$.

(b) $z\sin(1/z)$.

(c) $f(z) = \dfrac{\log z}{(z-1)^4}$ where $\log z$ is the principle branch of the logarithm defined in $\mathbb{P} = \{z : \mathrm{Re}\, z > 0\}$ (so that $\log 1 = 0$).