4 Functions and Harmonic sets

4.1 Functions of the form $(ax+b)/(cx+d)$

The purpose of this exercise is to complete the proof that the cross ratio $(x_1, x_2; x_3, x_4)$ is invariant under the map f defined by

$$f : x \mapsto (ax+b)/(cx+d)$$

provided $ad - bc \neq 0$.

It is quite easy to see that three special cases of this form have the property that the cross ratio is invariant under their application. They are the functions $x \mapsto x+k$, $x \mapsto mx, m \neq 0$, and $x \mapsto 1/x$. If we can show that f is a composition of functions of this form, then we will know that the cross ration is invariant under f. That is, that

$$(f(x_1), f(x_2); f(x_3), f(x_4)) = (x_1, x_2; x_3, x_4).$$

Let four numbers $a, b, c, d \in \mathbb{R}$ be given so that $ad - bc \neq 0$ and let $\mathbb{R}^* = \mathbb{R} \cup \{\infty\}$. Suppose we are given a map $f : \mathbb{R}^* \to \mathbb{R}^*$ defined by $x \mapsto (ax+b)/(cx+d)$.

4.1.1 Composition

Show that f may be expressed as a composition of a sequence of maps, f_1, f_2, ..., f_n so that each has one of these forms

$$\begin{array}{lcll} x & \mapsto & x+k & \\ x & \mapsto & mx, & \text{where } m \neq 0 \\ x & \mapsto & 1/x & \end{array}$$

and $f(x) = f_n(\dots f_2(f_1(x)) \dots)$.

4.1.2 ∞

For the function f what is the image of ∞? (Find y so that $y = f(\infty)$.) What is the pre-image of ∞? (Find x so that $f(x) = \infty$.)

4.1.3 Invertibility

For the numbers a, b, c and d, are there any conditions or special cases for which your sequence of functions is not invertible? What are they? Can any of the conditions be overcome by dealing with special cases? How?

4.2 Harmonic Sets

4.2.1 Experience

Not to hand in. Get experience points.

Let A, B, and C be any three points on a line L. Develop a GeoGebra figure that constructs the harmonic conjugate of C with respect to A and B. In this figure you will have to use some point that are not on L. Convince yourself (by moving them around) that your construction is independent of the position of any points you used that are not on the line L. Explore various arrangements of A, B, and C. Consider (i) C to the left of both A and B, (ii) C between A and B (iii) C to the right of both A and B. Do similar tests for B relative to A and C, and so on.) To sharpen your intuition learn for yourself about how D lies relative to A, B and C.

Experiment with your figure to find what happens to D if C is

1. very close to A.
2. very close to B.
3. at the midpoint of A and B.
4. near the midpoint of A and B.
5. very far away from A and B.
6. the ideal point on L.
7. Pick a position of C and notice where D is. What happens to D if you move C to where D was?

Swap the positions of A and B and repeat the above 7 experiments.

4.2.2 Reversability of harmonic conjugates

Let A, B, and C be any three distinct points. Prove that these two statements are equivalent:

- D is the harmonic conjugate of C with respect to A and B.
- C is the harmonic conjugate of D with respect to A and B.

4.2.3 Two parts of the projective line

In the Euclidean plane we think of two points on a line as separating the line into three parts, two rays and a segment. The the segment is, of course, the segment between the two points. Each ray starts at one of the points and goes in the direction away from the segment.

In the real projective line two points A and B separate the projective line into only two parts. one containing the point at infinity, the other not. Is it ever the case that both C and D are in the same part? Give a proof for your answer.

Hints:
(i) Consider the sign of $(a - x)/(x - b)$, including the sign when $x = \infty$.
(ii) Consider the definition of harmonic conjugates.

4.3 Harmonic sets are everywhere

4.3.1 The playing field

Imagine a rectangular playing field on a large open plane. Imagine a photograph of the field taken from a high point with a direction and an angle so that in the photograph both the field and the horizon are visible and so that, in the photo, the lines of the parallel sides of the field when extended converge on the line that is the image of the horizon.

Imagine, in the photo, two more lines that would correspond to the diagonals of the field. Each line joins one corner to its opposite corner. Imagine these

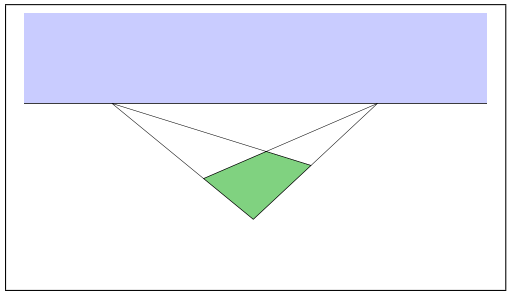

Figure 1: A rectangular playing field

lines also extended to the horizon, just like the other four lines that are the parallel sides of the field.

Show that these six lines from the field meet the horizon in four points that form a harmonic set of points. **Submit** a sketch, hand drawn, if you wish. (Artists understood vanishing points and perspective drawings before mathematicians formalized the study of projective geometry).

4.3.2 The general quadrangle

Recall that a frame of reference in the plane consists of four points, no three of them collinear. Thus they determine six distinct lines when taken two at a time. Show that, given any frame of reference in the plane, there are three lines $h_i, i = 1, 2, 3$ such each of them meets the six lines determined by the frame in a harmonic set of points.