7 Constructing Inverses

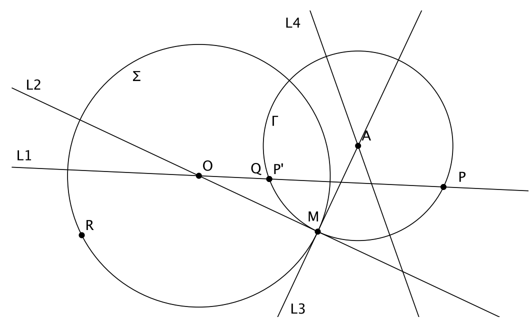
7.1 The inverse of a point

7.1.1 Construction using an orthogonal circle

Let O and R be distinct points and Σ be a circle with centre O and radius point R. Here is a construction for the inverse of P with respect to Σ.

```
L1 = line ( O, P );
M  = point ( Σ \ L1 );[a]
L2 = line ( O, M );
L3 = perp ( M, L2 );
L4 = pbis ( P, M );
A  = point ( L3, L4 );
Γ  = circle ( A, P );
Q  = point ( L1, Γ \ P ).
```

[a]The backslash may be read as "without" or as "but not on".



7.1.2 Exploration

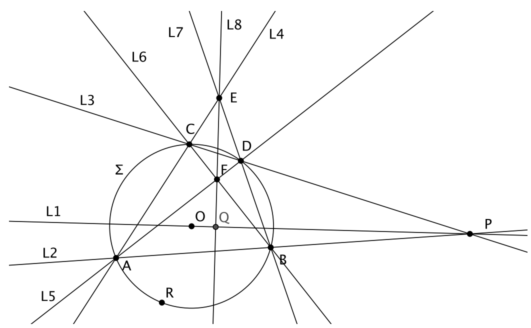
Confirm for yourself that when P is outside Σ, Q is inside, and vice versa. Confirm for yourself that Q seems not to move as M moves around Σ . Use the tool "Reflect Point about Circle" to find P′, the inverse of P with respect to Σ. Use the "Move" tool to shift the *labels* of P′, and Q slightly so that they do not overlap. Move the point P about and watch to see that P′, and Q always coincide. Use the tool "Relation between Two Objects" to compare P′, and Q. When using this tool for two points that overlap, it is easy to select the points by clicking on their labels.

7.2 A straightedge only construction

7.2.1 No compass needed here.

On a new figure, again let Σ be a circle with centre O, and let P be any point distinct from O. Construct Q as follows.

```
L1 = line ( O, P );
A  = point ( Σ );
L2 = line ( P, A ) ;
B  = point ( Σ, L2 \ A ) ;
C  = point ( Σ \ L2 ) ;
L3 = line ( P, C ) ;
D  = point ( Σ, L3 \ C ) ;
L4 = line ( A, C ) ;
L5 = line ( A, D ) ;
L6 = line ( B, C ) ;
L7 = line ( B, D ) ;
E  = point ( L4, L7 ) ;
F  = point ( L5, L6 ) ;
L8 = line ( E, F ) ;
Q  = point ( L1, L8 ).
```



7.2.2 Exploration

Confirm for yourself that the position of Q, the result of our construction, does not change as we move A or B anywhere on Σ, as long as the lines L2 and L3 each meet Σ in two distinct point.

Confirm for yourself that likewise, the position of the line L8 does not depend on the positions of A or B.

7.2.3 Comment

The circle is special case of a conic, and this construction makes of a polar line with respect to a conic (the circle). The line L8 is the polar line of the point P with respect to the circle Σ.

7.3 A self inverse construction

7.3.1 The best (!?) construction.

On a new figure, again let Σ be a circle with centre O, and let P be any point distinct from O. Construct Q as follows.

```
L1 = line ( O, P );
L2 = perp ( O, L1 );
A  = point ( L2, Σ );
B  = point ( L2, Σ \ A );
L3 = line ( A, P );
L4 = perp ( B, L3 );
Q  = point ( L4, L1 ).
```

7.3.2 Exploration

Experiment with the location of P, inside and outside Σ , to experience and become familiar with various qualitative properties of inversion: when P is inside, Q is outside, and the other way around. Also convince yourself that as P gets closer to O, Q gets much, much farther away, and the other way around. Also confirm that when P is on Σ, so is Q.

7.3.3 Comment

This construction might be the shortest one, both in terms of the number of steps used to define it and in terms of its computational complexity.

7.3.4 The invertibility / reversibility of this construction

Consider the positions of P and Q and the points and lines needed for the construction. Imagine now that you start your construction with Q instead of with P, and see that you can perform the steps listed above without adding any new points or new lines.

If we had *defined* the inverse of P to be Q as determined by this construction, above argument shows that the inverse of the inverse of P is P, itself.

7.3.5 A detail *

Considering the construction given in 7.3.1, recall that L4 was constructed to be perpendicular to L3 and passing through B. Let D be the point of intersection of the two lines L3 and L4. Notice in the figure that D appear to be on the circle Σ.

Use what you know about angles in a circle to prove that D is on Σ. (*)

7.3.6 Create a Tool

Use the figure you created in 7.3.1 to define a "Tool" or macro called **Inverse point** that uses a point and a circle as input objects and gives the inverse of a point with respect to the circle as the result.

Experiment with a new point R, and use your macro to construct R', the inverse of R with respect to Σ . Verify that when R is close to P, R' is close to Q, and when R is close to Q, R' is close to P.

7.4 The inverse of a line

7.4.1 Construction *

- On a new, blank figure construct a circle C with centre O.
- Let A and B be two new points anywhere, but not at O, and not on C.
- Construct the line "AB" defined by A and B.
- Use the "Parallel" tool to construct the line "b" through O parallel to the line AB.
- Let X be any point on the line AB.
- Use the built in tool "Reflect Point about Circle" to find X', the inverse of X with respect to the circle C.
- Use the Locus tool to find the locus of X' as a function of X.
- Notice that as you move A and B, the line b moves and the locus moves.

- Notice that the locus appears to be a circle tangent to m at O. Submit your figure together with its Construction Protocol. (*)

7.4.2 Observations

Again move A and B about and observe how the Locus moves. In particular, watch cases for which line AB meets the circle C in 0, 1 or 2 points. Include cases for which line AB is far from the circle, and for which AB is very near O, the centre of C. Notice that the point Y moves along the locus as you move X on the line AB.

7.5 The inverse of a circle

7.5.1 Construction of Locus (*)

In this item we work with the inverse of a circle C_2, with respect to a given circle Σ. Our first effort is to consider all the points X on C_2 and to find their inverses with respect to *Sigma*.

On a new figure, construct a circle Σ with centre O. Let C_2 be another circle anywhere in the plane, as long as it does not pass through O.

Let X be any point on C_2. Use the tool "Reflect Point about Circle" to construct the point Y, the inverse of X with respect to Σ. Constuct the locus of Y as a function of X. Submit your figure, together with its Construction Protocol. (*)

7.5.2 Observations (*)

Explore what happens as C_2 is adjusted to various locations and sizes. Consider locations for which C_2 is:

1. remote from, and outside of, Σ.
2. close to Σ
3. tangent to Σ,

4. meets Σ in two points,
5. is inside Σ,
6. is very close to O,
7. is on O,
8. is concentric with Σ (that is, Σ and C_2 have the same center).

Summarize your observations in items 3, 4 and 7 in 60 words or less. (*)

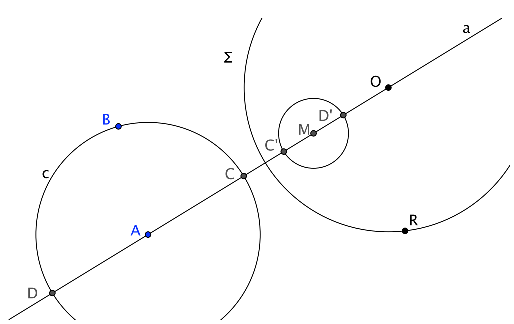
7.5.3 An algorithm for the inverse of a circle

The locus tool is not a completely satisfactory solution because it requires the use of the inverse algorithm for a large number, maybe hundreds of points. Your mission, if you choose to accept it, is this:

Given a circle Σ with center O, and a second circle C_2 with centre O_2, devise an algorithm for finding the inverse of C_2 with respect to Σ.

(When you get it right, you will be able to confirm that your algorithm gives the correct by comparing it with your locus result.)

```
Σ = circle ( O, R );
c  = circle ( A, B ) ;
a  = line (O, A) ;
A = point ( c, a ) ;
B = point ( c, a \ A ) ;
A' = inv ( A, Σ ) ;
B'  = inv ( A, Σ ) ;
M  = midpt ( A', B' ) ;
c' = circle ( M, D' ).
```



(*) Items marked with an asterisk should be handed in.