8 Pencils of circles
Poincaré model of Hyperbolic Plane

This set of questions is intended to give you experience with the three kinds of one parameter families[1]

(pencils) of circles. There are also examples of mutually orthogonal pencils of circles. In 8.2 you will be studying two pencils of circles in which each circle in one pencil is orthogonal to every circle in the other. In questions 8.3 and 8.4, ideas about circles in the Euclidean plane are applied to the Poincaré model of the real hyperbolic plane. Questions 8.5 and 8.6 deal again with families of circles in the Euclidean plane.

8.1 A family of circles

Devise a construction for the inverse of one circle with respect to a second. Hide the extra objects you used in the construction including the centre of the third circle. (Use this to create a Tool that gives the inverse of one circle with respect to the second.) On a fresh figure, start with two disjoint circles, H4 and H5 having about, but not exactly, the same size. Use the Tool to construct the following objects:

- H3, the inverse of H5 with respect to H4 ;
- H2, the inverse of H4 with respect to H3 ;
- H1, the inverse of H3 with respect to H2 .

Then construct:

- H6, the inverse of H4 with respect to H5 ;
- H7, the inverse of H5 with respect to H6 ;
- H8, the inverse of H6 with respect to H7 .

[1] A one parameter family is called **pencil**. The set of all points on a line is said to be the **pencil of points** on the line, because the points might be drawn by a pencil. This usage is extended to the pencil of lines on a point referring to the set of lines in the plane that are on a point, the set of planes on a line in three space, and any other one parameter family of objects.

Don't get carried away, you don't need more than this to learn what there is to be learned here.

Study the pencil (family) of circles as you move circles H4 and H5 to various positions with respect to each other. Watch the family as they change from being disjoint circles to tangent and then to overlapping circles. When the circles are disjoint, they are said to be members of **hyperbolic pencil**, when the circles form a tangent family, a **parabolic pencil**, and they form an overlapping family, an **elliptic pencil**. Notice that in the first category, no two circles meet; in the second category, all circles are tangent to each other the same point, an in the third category, there are two distinct points that are on all the circles in the pencil.

- Submit three figures, each illustrating one of the three types of pencils of circles. (*)
- Submit one copy of the Construction Protocol. (*)

8.2 Two Families of circles

Let C1 and C2 be two circles, with distinct centres o1 and o2, respectively.

8.2.1 D(x) and E(x)

Let x be a point other than `o1` and `o2`.

- Build circle D(x) and its centre d(x) as follows:

```
x1 = inv(x, C1) ;[2]
x2 = inv(x, C2) ;
d(x) = center ( x, x1, x2 ) ;[3]
D(x) = circle ( d(x), x ) ;
```

[2]The inverse of point `x` with respect to circle `C1`.
[3]The point `d(x)` is the centre of the circle on the named points.

- Build E(x) and its centre e(x) as follows.

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L1 = line ( o1, o2 ) ;
L2 = line ( d(x), x ) ;
L3 = perp ( x, L2 ) ;
e(x) = point ( L1, L3 ) ;
E(x) = circle ( e(x), x ) .
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8.2.2 Confirm orthogonality

Explain to your own satisfaction why the circles D(x) and E(x) are orthogonal to each other.

8.2.3 Build a Tool for E(x) and D(x)

Keep x, C1, C2 , d(x) and D(x), e(x), E(x) visible, but hide any other construction objects in your figure. Devise a Tool with input objects C1, C2 and x and output objects d(x), D(x), e(x) and E(x).

8.2.4 Build some examples

Use your Tool to construct several pairs of circles, each time using C1 and C2, but for various choices of x. Do not get carried away here, you don't need a lot to learn what you need to learn. Maybe 4 or 5 invocations of the Tool (8 or 10 circles) should be enough. Spread them around by arranging the positions of the various "x" points.

Warm up questions to answer for yourself:

See that there are two pencils of circles. One pencil given by the circles E(x) for any given x, and the other by circles D(x) for any given x.

Aside from the problem that the above construction might break if x lies on C1 or C2, convince yourself that C1 and C2 are members of the pencil of circles of the form E(x) of corresponding to positions of x that lie on C1 or C2.

Watch what happens to the two families of circles as you move C1 and C2 to various positions so that C1 and C2 meet in 0, 1 or 2 points.

1. For the case that circles C1 and C2 are disjoint:
   (a) What sort of family is the collection of all the circles D(x)?
   (b) What sort of family is the collection of all the circles E(x)?
   (c) Describe the locus of d(x), as x ranges over all possible points in the plane? (Are there any obvious omissions in the possible positions of d(x)?)
   (d) Describe the locus of e(x), as x ranges over all possible points in the plane? (Are there any obvious omissions in the possible positions of e(x)?)
   (e) Are there any points that are common to all the circles D(x)?
   (f) If so, is there a connection between them and the locus of e(x)?
   (g) Are there any points that are common to all the circles E(x)?
   (h) If so, is there a connection between them and the locus of d(x)?
2. The same as above, but the for the case in which C1 and C2 are tangent to each other.
3. The same as above, but for the case in which C1 and C2 meet in two distinct points.

8.2.5 Orthogonal families *

Start with a clean figure with non-intersecting circles C1 and C2, and two points r and s. Use two applications of the macro you devised in 8.2.3 to create the four circles, D(r), E(r), D(s) and E(s).

- Print your figure and your Construction Protocol. Confirm for yourself that D(r) and D(s) seem to meet at two points, u and v, on the line of centres of C1 and C2.
- Assuming the previous item, or otherwise, Explain why (prove) u and v are inverses of each other with respect to C1. *

- Using this fact, or otherwise, explain why (prove) D(r) and E(s) are orthogonal for any choice of r and s. *
- There may be some exceptional cases. If so, identify them.

8.3 Inverses of Families of Circles *

Let Σ be a circle whose centre is O and radius is r. Let A be any point interior to Σ. Suppose that A $\neq$ O. Let B = inv(A, Σ).

Let $\mathcal{F}$ be the family of circles that are incident with A and B. The circles in this family all have their centers on the line `pbis(A,B)`, the perpendicular bisector between A and B. Answer these 5 questions. *

1. Construct a circle Γ with centre at B that is orthogonal to Σ. Let s be defined to be the radius of Γ. Give your construction protocol.
2. Show that O and A are inverses of each other with respect to Γ.
   Hint: Use an equation implied by the fact that Γ is orthogonal to Σ.
3. Using the fact that all the circles in $\mathcal{F}$ go through the centre of Γ. what can you say about $\mathcal{F}^\Gamma$, the set of all circles that are inverses of circles in $\mathcal{F}$?
4. Consider inversion with respect to Γ. Let p be any point. Prove that:
   - If p is on Σ, then p^Γ is on Σ.
   - If p is inside Σ, then p^Γ is inside Σ.
   - If p is outside Σ, then p^Γ is outside Σ.

8.4 Reflections in the hyperbolic plane

In what follows, we are working in the Poincaré model of the hyperbolic plane.

Let Ω be any circle. The points and lines of the Poincaré model plane will be called P-points and P-lines to distinguish them from ordinary points and lines in the Euclidean plane. The **P-points** and **P-lines** are defined as follows.

- Any point inside (but not *on*) Ω is said to be a **P-point**.
- The interior arc of any circle orthogonal to Ω whose end points are on Ω is said to be a **P-line**. This includes the arcs of circles of infinite radius that are orthogonal to Ω, which happen to be diameters of Ω.

An inversion with respect to a P-line, when restricted to the P-points and P-lines of the model, is called a **P-reflection** or a reflection of the Poincaré model of the hyperbolic plane. If the P-line is an arc of a circle, the map is an inversion with respect to that circle restricted to the P-points. If the P-line is a segment, the map is a reflection with respect to that diameter of Ω.

8.4.1 Lemma *

Using the result in 8.3 (or otherwise), prove this lemma about the real hyperbolic plane:
Lemma 1. *Let O be the centre of Ω. Given any other P-point X, there is a P-line L, so that the P-reflection defined by L maps X to O and O to X. **

8.5 Elliptic pencil and lines on a point *

Using the construction in question 8.3 (or otherwise), explain how to invert any elliptic pencil of circles to a pencil of lines on a point. *

8.6 Hyperbolic pencil and concentric circles *

Using the ideas used in the construction in 8.3 (or otherwise), explain how to invert any hyperbolic pencil of circles to a pencil of concentric circles. *

(*) Items marked with an asterisk should be submitted for marking.