Pure Math 450/650, Assignment 1

Due: January 13.

1. Let $a < b$ in $\mathbb{R}$, $\mathcal{X}$ be a Banach space and $f : [a, b] \to \mathcal{X}$ be a function.

   (a) Prove that if $f : [a, b] \to \mathcal{X}$ is *piecewise continuous* [i.e. there is a partition $\{a = a_0 < a_1 < \cdots < a_k = b\}$ such that, f is continuous on each interval (a_{i-1}, a_i)], and *bounded* [i.e. $\sup_{t\in[a,b]} \|f(t)\| < \infty$], then it is Riemann integrable.

   [Hint: You may use, without proof, the fact that a continuous function on a compact set is uniformly continuous.]

   (b) Show that if $f : [a, b] \to \mathcal{X}$ and $\|f(\cdot)\| : [a, b] \to \mathbb{R}$ are both Riemann integrable, then

   $$\left\| \int_a^b f(t)dt \right\| \le \int_a^b \|f(t)\| \, dt.$$

   Note that if f is piecewise continuous and bounded, then so too is $\|f(\cdot)\|$.

   (c) (BONUS) Determine whether having $f : [a, b] \to \mathcal{X}$ Riemann integrable implies that $\|f(\cdot)\| : [a, b] \to \mathbb{R}$ is Riemann integrable

2. Let $|X|$ denote the *cardinality* of a set X. Show that $|\mathbb{R}| = |\{0, 1\}^{\mathbb{N}}|$, where $\{0, 1\}^{\mathbb{N}} = \big\{(\varepsilon_1, \varepsilon_2, \dots) : \varepsilon_i \in \{0, 1\} \text{ for } i \text{ in } \mathbb{N}\big\}$, by establishing an *explicit* bijection $\varphi : \mathbb{R} \to \{0, 1\}^{\mathbb{N}}$.

   [Hint: Composition of maps is allowed, Cantor-Bernstein is not. Try representing elements of $(0, 1)$ in binary, rather than decimal form.]

3. Show that the set of rational numbers, $\mathbb{Q}$, cannot be realized as the intersection of a *sequence* of open subsets of $\mathbb{R}$. (In other words, $\mathbb{Q}$ is not a G_δ *set*.)

   [Hint: Baire.]

4. **(a)** Show that if G is an open set in $\mathbb{R}$, then there exists a sequence $J_1, J_2, \dots$ of open intervals (which may be a finite sequence) such that

   $$\text{(i)} \;\; J_i \cap J_j = \varnothing \text{ if } i \neq j, \text{ and}$$

   $$\text{(ii)} \;\; G = \bigcup_{i=1,2,\dots} J_i$$

   Moreover, this sequence is unique up to re-indexing.

   Thus for, for G and $J_1, J_2, \dots$ above, it follows from the σ-additivity of λ that

   $$\lambda(G) = \sum_{i=1,2,\dots} \ell(J_i).$$

   (b) Deduce that for any set $E \subset \mathbb{R}$, we have

   $$\lambda^*(E) = \inf\{\lambda(G) : E \subset G, G \text{ is open}\}.$$