**Pure Math 450/650, Assignment 1: Sketch of Solutions**

1. **(a) Step #1.** Suppose that $f$ is continuous except possibly at the eanpoints $a$ and $b$. Let $M = \sup_{t\in[0,1]} \|f(t)\|$ which we may suppose to be non-zero. Let $\varepsilon > 0$ be given. Let
$$a_1 \in \left(a, a + \frac{\varepsilon}{8M}\right), b_1 \in \left(b - \frac{\varepsilon}{8M}, b\right)$$
be choosen so $a_1 < b_1$. Then $f$ is continuous on the compact interval $[a_1, b_1]$. Let $\delta > 0$ be choosen so for $s, t$ in $[a_1, b_1]$ with $|s - t| < \delta$ we have $\|f(s) - f(t)\| < \frac{\varepsilon}{2(b-a)}$. Now choose $a = a_0 < a_1 < a_2 < \cdots < a_{n-2} < a_{n-1} = b_1 < a_n = b$ so
$$a_k - a_{k-1} < \delta \text{ for } k = 2, \dots, n-1$$
and let $\mathcal{P}_\varepsilon = \{a_0, \dots, a_n\}$. Now consider partitions of $[a, b]$ $\mathcal{P} = \{b_0 < \cdots < b_{n_1}\}$, $\mathcal{Q} = \{c_0 < \cdots < c_{n_2}\} \supset \mathcal{P}_\varepsilon$ and consider Riemann sums
$$S(f, \mathcal{P}) = \sum_{i=1}^{n_1} f(s_i^*)(b_i - b_{i-1}) \text{ and } S(f\mathcal{Q}) = \sum_{j=1}^{n_2} f(t_j^*)(c_j - c_{j-1}).$$
Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q} = \{d_0 < \cdots < d_m\}$. For each $k = 1, \dots, m$ let
$$\begin{aligned} u_k^* &= s_i^* \text{ where } [d_{k-1}, d_k] \subset [c_{i-1}, c_i] \text{ and} \\ u_k^{**} &= t_j^* \text{ where } [d_{k-1}, d_k] \subset [d_{j-1}, d_j]. \end{aligned}$$
Then we have
$$\begin{aligned} \|S(f, \mathcal{P}) - S(f, \mathcal{Q})\| &= \left\| \sum_{k=1}^{m} f(u_k^*)(d_k - d_{k-1}) - \sum_{k=1}^{m} f(u_k^{**})(d_k - d_{k-1}) \right\| \\ &\le \sum_{k=1}^{m} \|f(u_k^*) - f(u_k^{**})\| (d_k - d_{k-1}) \\ &\le \sum_{d_k \le a_1 \text{ or } d_{k-1} \ge a_{n-1}} (\|f(u_k^*)\| + \|f(u_k^{**})\|)(d_k - d_{k-1}) \\ &\quad + \sum_{a_1 \le d_{k-1} < d_k \le a_{n-1}} \|f(u_k^*) - f(u_k^{**})\| (d_k - d_{k-1}) \\ &< 2M(a_1 - a_0 + a_n - a_{n-1}) + \frac{\varepsilon}{2(b-a)}(a_{n-1} - a_1) < \varepsilon. \end{aligned}$$
[Indeed, by choice of $u_k^*, u_k^{**}$ we have that they are both in the same interval of the partition $\mathcal{P}_\varepsilon$, and when that interval is within $[a_1, b_1]$ we have $|u_k^* - u_k^{**}| < \delta$.] Hence by the Cauchy Criterion for Riemann integrability we are done.

**Step #2.** If $a = a_0 < a_1 < \cdots < a_{n-1} < a_m = b$ represent all the possible points of discontinuity we have, from above, that $f$ is Riemann intgerable on each interval

$[a_{k-1}, a_k]$, $k = 1, \dots, n$. Given $\varepsilon > 0$, find for each $k-1, \dots n$ a partition $\mathcal{P}_k$ of $[a_{k-1}, a_k]$ such that

$$\left\| S(f, \mathcal{Q}_k) - \int_{a_{k-1}}^{a_k} f \right\| < \frac{\varepsilon}{n} \text{ whenever } \mathcal{Q}_k \supset \mathcal{P}_k$$

for any Riemann sum $S(f, \mathcal{Q}_k)$. Now if $\mathcal{P} = \bigcup_{k-1}^{n} \mathcal{P}_k$ then for any partition $\mathcal{Q} \supset \mathcal{P}$ we can write $\mathcal{Q}_k = \{b \in \mathcal{Q} : a_{k-1} \leq b \leq a_k$ and we can decompose $S(f, \mathcal{Q}) = \sum_{k=1}^{n} S(f, \mathcal{Q}_k)$ as a sum of Riemann sums over subintervals. We thus have

$$\left\| S(f, \mathcal{Q}) - \sum_{k=1}^{n} \int_{a_{k-1}}^{a_k} f \right\| \leq \sum_{k=1}^{n} \left\| S(f, \mathcal{Q}_k) - \int_{a_{k-1}}^{a_k} f \right\| < \varepsilon.$$

**(b)** It is obvious that $\|S(f, \mathcal{P})\| \leq S(\|f\|, \mathcal{P})$. If the desired inequality failed to hold, let $\varepsilon = \left\| \int_a^b f \right\| - \int_a^b \|f\|$, and show this leads to a contradiction using a partition $\mathcal{P}$ for which $\left| S(\|f\|, \mathcal{P}) - \int_a^b \|f\| \right| < \varepsilon/2$ and $\left\| S(f, \mathcal{P}) - \left\| \int_a^b f \right\| \right\| < \varepsilon/2$, and a clever use of subadditivity of the norm.

2. Define $\tau : (0,1) \to \mathbb{R}$ by $\tau(x) = 1/(x^2 - x)$. Verify this is a bijection and let $\varphi_1 = \tau^{-1}$. Let $\{t_n\}_{n=1}^{\infty}$,$\{s_n\}_{n=1}^{\infty}$ be disjoint sequences of distinct points in $(0,1)$ (find examples). Define a $\psi : (0,1) \to (0,1) \setminus \{t_n\}_{n=1}^{\infty}$ by

$$\psi(t) = \begin{cases} t & \text{if } t \notin \{t_n, s_n\}_{n=1}^{\infty} \\ s_{2n} & \text{if } t = t_n \\ s_{2n-1} & \text{if } t = s_n. \end{cases}$$

Verify that $\psi$ is a bijection.

Every element $t$ in $(0,1)$ ha s a unique representation in binary $t = 0.\varepsilon_1\varepsilon_2 \cdots := \sum_{k=1}^{\infty} \varepsilon_k/2^k$, where each $\varepsilon_k \in \{0,1\}$ provided we don't allow sequences which are ultimately constant 1. Let $x_1, x_2, \dots$ be the enumaration of

$$\begin{gathered} (0,0,0,\dots), (1,1,1,\dots), \\ (0,1,1,1,\dots), \\ (0,0,1,1,1,\dots), (1,0,1,1,1,\dots), \\ (0,0,0,1,1,\dots), (0,1,0,1,1,\dots), (1,0,0,1,1,\dots), (1,1,0,1,1,\dots), \\ \vdots \end{gathered}$$

the sequence of elements which are ultimately constant 1, as suggested. Define $\theta : \{0,1\}^{\mathbb{N}} \to (0,1)$ by

$$\theta(\varepsilon_1, \varepsilon_2, \dots) = \begin{cases} \psi(0, \varepsilon_1\varepsilon_2 \dots) & \text{if } (\varepsilon_1, \varepsilon_2, \dots) \notin \{x_n\}_{n=1}^{\infty} \\ t_n & \text{if } (\varepsilon_1, \varepsilon_2, \dots) = x_n. \end{cases}$$

Verify that $\theta$ is a bijection so $\varphi_2 = \theta^{-1} : (0,1) \to \{0,1\}^{\mathbb{N}}$ is a bijection.

Let $\varphi = \varphi_2 \circ \varphi_1$.

3. Suppose $\mathbb{Q} = \bigcap_{n=1}^{\infty} G_n$, $G_n$ open. Write $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$. Then each $F_n = (\mathbb{R} \setminus G_n) \cup \{r_n\}$ is closed and nowhere dense. (Why?) But then $\mathbb{R} = \bigcup_{n=1}^{\infty} F_n$.

4. **(a)** On $G$ define a relation

$$x \sim y \quad \Leftrightarrow \quad \text{there are } a < b \text{ s.t. } x, y \in (a,b) \subset G$$

for $x, y$ in $G$. Verify that this is an equivalence relation. Suppose the equivalence class $[x]$ is bounded. Then for $0 < \varepsilon < \frac{\sup[x]-\inf[x]}{2}$, we have $\inf[x] + \varepsilon, \sup[x] - \varepsilon \in [x]$ and hence $(\inf[x] + \varepsilon, \sup[x] - \varepsilon) \subset [x]$. Thus

$$[x] \supset \bigcup_{0<\varepsilon<\frac{\sup[x]-\inf[x]}{2}} (\inf[x] + \varepsilon, \sup[x] - \varepsilon) = (\inf[x], \sup[x]).$$

We have $[x] \subset (\inf[x], \sup[x])$: for if we had $y \in [x]$ for any $y \leq \inf[x]$, or $y \geq \sup[x]$, this would violate the definition of $y \sim x$. Thus $[x] = (\inf[x], \sup[x])$. An unbounded equivalence class can be dealt with similarly and will yield a half-infinite interval. The equivalence classes partition $G$ in a unique manner.

Let $\mathbb{Q} \cap G = \{r_k\}_{k=1}^{\infty}$. For $x$ in $G$ let $k_x = \min\{k : q_k \in [x]\}$, and write $I_{k_x} = [x]$. (Also, an axiom of choice argument allows a choice of one $q_x \in \mathbb{Q} \cap [x]$ for each $x$ in $G$; write $I_{q_x} = [x]$.) Thus the collection if intervals $\{I_{k_x}\}_{x \in G}$ (or $\{I_{q_x}\}_{x \in G}$) is countable; enumerate as $\{J_n\}_{n=1,2,\ldots}$

**Alternate:** Let $(X, \rho)$ be a metric space with the property such that each open ball $B(x_0, r) = \{x \in X : \rho(x, x_0) < r\}$ is connected, i.e. if open sets $U, V \subset X$ satisfy $B(x_0, r) \subset U \cup V$ then $U \cap V \neq \varnothing$. [Exercise: Find a metric space without this property.] Let $G \subset X$ be open. Define an relation on $G$ by

$$x \sim y \quad \Leftrightarrow \quad \text{there is a connected } C \text{ with } x, y \in C \subset G.$$

Then it is straightforward to verify that $\sim$ is an equivalence relation; transitivity is the only slightly tricky part. Note that $[x] = \bigcup\{C : x \in C \subset G \text{ and } C \text{ is connected}\}$ so $[x]$ is connected (needs verifying!). Moreover, since $G$ is open and all balls are open, it follows that $[x]$ contains a neighbourhood of each of its points, i.e. is open. We have $G$ is partitioned uniquely by these open equivalence classes, i.e. $G = \dot\bigcup_{V \in G/\sim} V$.

If we further assume that $X$ is separable, i.e. it admits a dense sequence $\{d_j\}_{j=1}^{n}$, then, as above, we can show that there are, at most, countably many open equivalence classes $V$.

It remains to show that open intervals are the only open connected subsets in $\mathbb{R}$. The proof of the intermediate value theorm of M147 can be adapted to show that any interval is connected. Indeed, any open connected set $V \subset \mathbb{R}$ satisfies the intermediate value property: if $x < y$ in $V$ and $x < z < y$ then $z \in V$ too; indeed otherwise $\{(-\infty, z), (z, \infty)\}$ forms a disconnection of $V$. Thus $V = \bigcup_{x<y \text{ in } V}(x, y) = (\inf V, \sup V)$ is an interval.

**(b)** We have that $\lambda^*(E) \leq \inf\{\lambda(G) : G \supset E, G \text{ is open}\}$, from $\sigma$-subadditivity of $\lambda^*$ and that $\lambda^*(E) \leq \lambda^*(G) = \lambda(G)$ if $E \subset G$. [Alternatively, $\inf\{\lambda(G) : G \supset E, G \text{ is open}\}$ represents the infemum taken over covers of $E$ by sequences of pairwise disjoint open intervals, and hence $\lambda^*(E) \leq \inf\{\lambda(G) : G \supset E, G \text{ is open}\}$ since there are clearly more covers of $E$ by sequences of open intervals which are not necessarlity pairwise disjoint.]

Conversely, if $\{I_n\}_{n=1}^{\infty}$ is a cover of $E$ by open intervals, then by increasing property and $\sigma$-subadditivity $\lambda^*(E) \leq \lambda^*(G) \leq \sum_{n=1}^{\infty} \lambda^*(I_n) = \sum_{n=1}^{\infty} \ell(I_n)$, where $G = \bigcup_{n=1}^{\infty} I_n$ is open.