Pure Math 450, Assignment 2: Solution Sketch

1. **(a)** The case $\lambda(E) = \infty$ is obvious. Otherwise, find for each n an open $G_n \supset E$ such that $\lambda(G_n) < \lambda^*(E) + 1/n$. Let $A = \bigcap_{n=1}^{\infty} G_n$.

   (b) Suppose E is measurable. Find measurable sets E_n so $\lambda(E_n) < \infty$ and $E = \bigcup_{n=1}^{\infty} E_n$ (say $E_n = E \cap (-n, n)$). Find open $G_{n,k} \supset E_n$ so $\lambda(G_{n,k}) < \lambda(E_n) + 1/(k2^n)$. Then $G_k = \bigcup_{n=1}^{\infty} G_{n,k}$ contains E with $\lambda(G_k \setminus E) < 1/k$. To verify this note that $G_k \setminus E = \bigcap_{n=1}^{\infty}(G_k \setminus E_n) \subset \bigcup_{n=1}^{\infty}(G_{n,k} \setminus E_n)$ [the last inclusion requires verification]. Then let $A = \bigcap_{k=1}^{\infty} G_k$ and verify that $\lambda(A \setminus E) = 0$.

   If there is any measurable set $A \supset E$ for which $\lambda^*(A \setminus E) = 0$ then $E = A \setminus (A \setminus E) = A \cap (\mathbb{R} \setminus (A \setminus E))$ is measurble too.

   (c) We verify " $\Leftarrow$ " only. Use part (a) to find measuable A so $E \subset A \subset B$ and $\lambda(A) = \lambda^*(E)$. Then

$$\begin{aligned} \lambda(B) &= \lambda^*(E) + \lambda^*(B \setminus E) \\ &= \lambda^*(E) + \big[\lambda^*((B \setminus E) \cap A) + \lambda^*((B \setminus E) \setminus A)\big] \\ &= \lambda^*(E) + \big[\lambda^*(A \setminus E) + \lambda^*(B \setminus A)\big] \\ &= \big[\lambda(A) + \lambda(B \setminus A)\big] + \lambda^*(A \setminus E) \\ &= \lambda(B) + \lambda^*(A \setminus E) \end{aligned}$$

   so $\lambda^*(A \setminus E) = 0$.

2. **(a)** If $K \subset E$ is compact then

$$\lambda(K) + \lambda^*(E) \leq \underbrace{\lambda(K) + \lambda((a,b) \setminus K)}_{\lambda((a,b))} \leq \lambda_*(E) + \lambda((a,b) \setminus K).$$

   Indeed $(a,b) \setminus K$ is open and contains $(a,b) \setminus E$; thus $\lambda^*(E) \leq \lambda((a,b) \setminus K)$. On the left take supremum over all such K; on the right take infemum over all such K (here we should be cautious that if $(a,b) \setminus E \subset G \subset (a,b)$ then we cannot gurantee that $(a,b) \setminus G$ is compact; however $[a+\varepsilon, b-\varepsilon] \setminus G$ is, take $\varepsilon \to 0$). In either case we obtain $\lambda_*(E) + \lambda^*((a,b) \setminus E)$.

   (b) If $E \subset (a,b)$, compare result in (a) with Caratheodory criterion. If E is unbounded, write $E = \dot{\bigcup}_{n=1}^{\infty} E_n$ where each E_n is measurable and bounded. How? Given $\varepsilon > 0$, for each n find compact $K_n \subset E_n$ so $\lambda(K_n) > \lambda(E_n) - \varepsilon/2^n$. Then for each m, $C_m = \bigcup_{n=1}^{m} K_n \subset E$ is compact. Verify that

$$\lim_{m\to\infty} \lambda(C_m) = \lim_{m\to\infty} \sum_{n=1}^{m} \lambda(K_n) \geq \lim_{m\to\infty} \sum_{n=1}^{m} \left(\lambda(E_n) - \frac{\varepsilon}{2^n}\right) = \lambda(E) - \varepsilon.$$

   Thus $\lambda_*(E) \geq \lambda(E)$ and we are done. Why?

(c) Use (a), and 1. (c), with $B = (a, b)$ being any interval containing A.

3. **(a)** Intersection of closed sets is closed.

   Say C_α contained a non-empty open interval (a, b). Then $(a, b) \subset C_{n,\alpha}$ for each n. Each $C_{n,\alpha}$ is a disjoint union of 2^{n+1} closed intervals, each of length no greater than $1/2^{n+1}$. Explain. Hence (a, b) must be contained such an interval. For large enough n this is absurd.

   (b) **Solution #1:** Let $I_{n,1}, \ldots, I_{n,2^n}$ denote the open intervals removed from $C_{n-1,\alpha}$ to make $C_{n,\alpha}$. Each has length $\alpha/3^n$. Then we use De Morgan's Law to verify

$$[0, 1] \setminus C_\alpha = \dot{\bigcup_{n=0}^{\infty}} \dot{\bigcup_{k=1}^{2^n}} I_{n,k} \text{ and hence } \lambda([0, 1] \setminus C_\alpha) = \sum_{n=0}^{\infty} \alpha \frac{2^n}{3^{n+1}} = \alpha.$$

   Thus $\lambda(C_\alpha) = 1 - \alpha$

   Solution #2: Each set $C_{\alpha,n}$ is a disjoint union of 2^n closed intervals, $J_{n,1}, \ldots, J_{n,2^n}$, each of the same length ℓ_n. By the definition of the $J_{n,k}$'s, we find that the sequece $\ell_0, \ell_1, \ell_2, \ldots$ satifies the recursion

$$\ell_{n+1} = \frac{1}{2}\left(\ell_n - \frac{\alpha}{3^{n+1}}\right), \quad \ell_0 = \lambda([0, 1]) = 1.$$

   One can check, by induction, that

$$\ell_n = \frac{1}{2^n} - \frac{\alpha}{2} \sum_{k=1}^{n} \frac{1}{2^{n-k}3^k}, \quad \text{for } n = 1, 2, \ldots$$

   Hence for $n = 1, 2, \ldots$ we have

$$\lambda(C_{\alpha,n}) = 2^n \ell_n = 1 - \frac{\alpha}{2} \sum_{k=1}^{n} \frac{2^k}{3^k} = 1 - \alpha(1 - (2/3)^{n+1}).$$

   Thus we see that $\lim_{n\to\infty} \lambda(C_{\alpha,n}) = 1 - \alpha$. We are done once we have the following result:

   Proposition: *Let $A_1 \supset A_2 \supset \cdots \in \mathcal{L}(\mathbb{R})$ be so $\lambda(A_1) < \infty$. Then*

$$\lambda\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \lambda(A_n).$$

   Proof. Verify first that $A_m = \dot{\bigcup}_{n=m}^{\infty}(A_n \setminus A_{n+1}) \dot{\cup} \bigcap_{n=1}^{\infty} A_n$. Thus

$$\lambda(A_m) = \sum_{n=m}^{\infty} \lambda(A_n \setminus A_{n+1}) + \lambda\left(\bigcap_{n=1}^{\infty} A_n\right).$$

   The above sum is the "tail end" of a converging series. Why? Take $m \to \infty$. $\square$

(c) Let $A = \bigcup_{m=1}^{\infty} C_{1/m}$. Verify that $\lambda(A) = 1$ and A is of first category. Let $B = [0,1] \setminus A$.

(d) Let for each n, $J^{\alpha}_{n,1}, \dots, J^{\alpha}_{n,2^{n+1}}$ denote the component intervals of $C_{\alpha,n}$; and $I^{\alpha}_{n,1}, \dots, I^{\alpha}_{n,2^{n+1}}$ their open middles of length $\frac{\alpha}{3^{n+1}}$ [thus $J^{\alpha}_0 = [0,1]$ and $I^{\alpha}_{01} = \left(\frac{1}{2} - \frac{\alpha}{6}, \frac{1}{2} + \frac{\alpha}{6}\right)$]. We similarly let $J_{n,1}, \dots, J_{n,2^{n+1}}$ and $I_{n,1}, \dots, I_{n,2^{n+1}}$ be such intervals for the "middle third", i.e. $\alpha = 1$, Cantor set.

Solution #1: For each n we define $\varphi_n : [0,1] \to [0,1]$ to be the unique piecewise linear, striclty increasing function such that
(i) $\varphi_n(J^{\alpha}_{n,k}) = J_{n,k}$ for $k = 1, \dots, 2^{n+1}$, and
(ii) $\varphi_n(I^{\alpha}_{m,k}) = I_{m,k}$ for $m = 0, 1, \dots, n$ and $k = 1, \dots, 2^{m+1}$.
Verify (tediously) that

$$|\varphi_n(x) - \varphi_n(y)| \leq \frac{1}{\alpha}|x - y| \text{ for each } x, y \in [0,1]$$

(that $1/\alpha \geq 1$ is helpful).

[At this point we may observe that $\{\varphi_n\}_{n=1}^{\infty}$ is equicontinuous (given $\varepsilon > 0$ let $\delta = \frac{\varepsilon}{\alpha}$), and trivially this family is bounded. Thus by Arzela-Ascoli, there is a converging sequence from within this set of functions — though not obviously a subsequece of the sequence $(\varphi_n)_{n=1}^{\infty}$! (How?) However, it is possible to verify that the created sequence contains a subsequence which is also a subsequence of $(\varphi_n)_{n=1}^{\infty}$. As below, verify that this subsequence converges to a strictly increasing φ on $[0,1]$ for which $\varphi(C_\alpha) = C$.]

Note that if $y \in I^{\alpha}_{N,k}$, then for any $n \geq N$, $\varphi_n(y) = \varphi_N(y)$. If $\varepsilon > 0$ is given, and $x \in C_\alpha$, then by part (a), there is $y \in [0,1] \setminus C_\alpha$ such that $|x - y| \leq \frac{\alpha\varepsilon}{2}$. Then $y \in I^{\alpha}_{N,k}$ for some N. Then if $n, m \geq N$ we have

$$\begin{aligned} |\varphi_n(x) - \varphi_m(x)| &\leq |\varphi_n(x) - \varphi_N(y)| + |\varphi_N(y) - \varphi_m(x)| \\ &= |\varphi_n(x) - \varphi_n(y)| + |\varphi_m(y) - \varphi_m(x)| \\ &\leq \frac{1}{\alpha}|x - y| + \frac{1}{\alpha}|x - y| < \varepsilon. \end{aligned}$$

Hence the sequence $\{\varphi_n\}_{n=1}^{\infty}$ is unifomly Cauchy; since the space $(\mathcal{C}[0,1], \|\cdot\|_\infty)$ is complete, the sequence has a uniform limit φ which must be continuous (PMath 351)

We have that φ is strictly increasing since if $x < y$ in $[0,1]$ we have that, by part (a), there is $x < z < y$ such that $z \in [0,1] \setminus C_\alpha$. Thus $z \in I^{\alpha}_{N,k}$ for some N and k, whence there is z' in $I^{\alpha}_{N,k}$ with $z' > z$. Thus

$$\varphi(x) \leq \varphi(z) = \varphi_N(z) < \varphi_N(z') = \varphi(z') \leq \varphi(y).$$

Thus φ is a continuous, increasing map, so it has continuous inverse. Also verify that $\varphi(C_\alpha) = C$ so $\varphi|_{C_\alpha}$ is a homoemorphism.

Solution #2: Let $I^\alpha_{n,k} = (a^\alpha_{n,k}, b^\alpha_{n,k})$ and $I_{n,k} = (a_{n,k}, b_{n,k})$ for all n, and $k = 1, \dots, 2^n$. Then, since by (a), $[0,1] \setminus C_\alpha = \bigcup_{n=0}^\infty \bigcup_{k=1}^{2^n} I^\alpha_{n,k}$ is dense in $[0,1]$, we see that $E_\alpha = \{a^\alpha_{n,k}, b^\alpha_{n,k} : k = 1, \dots, 2^n, n = 1, 2, \dots\}$ (E="endpoint set"), which is clearly in C_α, is dense in C_α. Similarly $E = \{a_{n,k}, b_{n,k} : k = 1, \dots, 2^n, n = 1, 2, \dots\}$ is dense in C. Define a map $\varphi_0 : E_\alpha \to E$ by $\varphi_0(x^\alpha_{n,k}) = x_{n,k}$ where x is either of the symbols a or b. We see that φ_0 is strictly increasing and that

$$|\varphi_0(x) - \varphi_0(y)| \leq \frac{1}{\alpha}|x - y|$$

for x, y in E; this inequality can be checked similarly to the inequality in the solution above. Thus φ_0 is Lipschitz, hence uniformly contnuous on a dense subset of C_α. Thus there exists a unique (uniformly) continuous $\varphi : C_\alpha \to C$ such that $\varphi|_{E_\alpha} = \varphi_0$ (PMath 351). Also if $x < y$ in C_α, use density of E_α and an induction argument to find sequences $x < \dots < x_2 < x_1 < y_1 < y_2 < \dots < y$ with $x_i, y_i \in E_\alpha$ and $|x - x_n|, |y - y_n| < 1/n$. Thus

$$\varphi(x) = \lim_{n\to\infty} \varphi_0(x_n) < \lim_{n\to\infty} \varphi_0(y_n) = \varphi(y).$$

Thus $\varphi : C_\alpha \to C$ is continuous and injective. Moreover $\varphi(C_\alpha)$ is a compact, hance closed subset of C, containg the dense set E, hence φ is surjective. By a PMath 351 result, a continuouis bijection between compact sets is necessarily a homeomorphism.

Solution #3: List the component interval of C_n in the following manner:

$$C_n = \dot{\bigcup_{a\in\{0,2\}^n}} I_a \text{ where } I_a = \{t \in [0,1] : t = 0.a_1 \dots a_n t_{n+1} t_{n+2} \dots \text{ (ternary)}\}.$$

Then we know from class that $t \in C$ if and only if $t = 0.t_1 t_2 \dots$ (ternary) with $t_i \in \{0, 2\}$. Moreover, this sequence of t_i's is uniquely detemined by

$$t = 0.t_1 t_2 \dots \quad \Leftrightarrow \quad t \in \bigcap_{n=1}^\infty I_{t_{(n)}} \text{ where } t_{(n)} = \{t_1, \dots, t_n\} \in \{0,2\}^n.$$

Now, fix $0 < \alpha < 1$, and denote the nth set in the constuction of C_α by $C_{\alpha,n}$. There are 2^n intervals, $J_a, a \in \{0,2\}^n$ such that $C_{\alpha,n} = \dot\bigcup_{a\in\{0,2\}^n} J_a$. Note that the indices are chosen to satisfy $\sup J_a < \inf J_b$ if and only if $a \prec b$ where $\prec$ is the lexicographical ordering on strings in $\{0,2\}^n$, i.e. $(0, \dots, 0) \prec (0, \dots, 0, 2) \prec \dots \prec (2, \dots, 2)$. Also, given $a \in \{0,2\}^n$, $J_{(a_1,\dots,a_n,0)}$ and $J_{(a_1,\dots,a_n,2)}$ are the distinct subintervals of J_a in the description of $C_{\alpha,n+1}$. We note that length$(J_a) \leq 1/2^n$ if $a \in \{0,2\}^n$, and each J_a is closed. Hence given $a \in \{0,2\}^{\mathbb{N}}$, $\bigcap_{n=1}^\infty J_{a_{(n)}} = \{x_a\}$ for a unique element x_a ($a_{(n)} = (a_1, \dots, a_n)$, nth truncation of a).

We let $\varphi : C \to C_\alpha$ by $\varphi(t) = x_{\tilde t}$, where $\tilde t = (t_1, t_2, \dots) \in \{0,2\}^{\mathbb{N}}$ is given by $t = 0.t_1 t_2 \dots$ (ternary, no 1's). The properties discussed about the intervals J_a

above tell us that φ is stricly increasing, and injective. It is also surjective as $C_\alpha = \bigcap_{n=1}^{\infty} C_{\alpha,n}$. If we can verify the continuity of φ, then it is automatically a homeomorphism (PMath 351). Let $\varepsilon > 0$ be given. Find n so $1/2^n < \varepsilon$. Now if $s, t \in C$ satisfy $|s - t| < 1/3^n$, then they must live in the same interval I_a in C_n. Thus $x_{\tilde{s}}$ and $x_{\tilde{t}}$ live in the same interval J_a in $C_{\alpha,n}$ (in fact $\tilde{s}_{(n)} = \tilde{t}_{(n)} = a$). Hence $|\varphi(s) - \varphi(t)| = |x_{\tilde{s}} - x_{\tilde{t}}| \leq 1/2^n < \varepsilon$. Hence (uniform) continuity of φ holds.

4. **(a)** Clearly $|\mathcal{L}(\mathbb{R})| \leq |\mathcal{P}(\mathbb{R})| = 2^c$. On the other hand we have that if C is the usual middle thirds Cantor set than $\mathcal{P}(C) \subset \mathcal{N}(\mathbb{R}) \subset \mathcal{L}(\mathbb{R})$ [$\mathcal{N}(\mathbb{R})$ is the family of null sets]. Hence $2^c = |\mathcal{P}(\mathbb{R})| = |\mathcal{P}(C) \leq |\mathcal{P}(\mathbb{R})|$. The Cantor-Bernstein Theorem tells us that $2^c = |\mathcal{L}(\mathbb{R})|$.