Pure Math 450, Assignment 3

Due: Friday, February 10.

1. Show that any bounded subset A of $\mathbb{R}$, such that $\lambda^*(A) > 0$, contains a nonmeasurable subset.

   [Hint: The proof of the existence of a nonmeasurable subset of $(-a, a)$, given in class, can be adapted.]

2. Let C denote the Cantor ternary set in $[0, 1]$. Recall that each t in $[0, 1]$ admits a ternary expansion

$$t = 0.t_1t_2\cdots = \sum_{i=1}^{\infty} \frac{t_i}{3^i}, \text{ where } t_i \in \{0, 1, 2\}.$$

   Recall too that $G = [0, 1] \setminus C$ consists of all elements in $[0, 1]$ whose ternary expansion must contain a 1. Define $\varphi : G \to [0, 1]$ by

$$\varphi(0.t_1t_2\ldots) = \frac{1}{2^n} \sum_{k=1}^{n} t_k{}^{n-k}, \text{ if } t_n = 1 \text{ and } t_\ell \neq 1 \text{ for any } \ell = 1, \ldots n-1.$$

   (a) Let

$$G_3 = \left\{ 0.t_1t_2t_3\cdots \in [0,1] : \begin{array}{r} t_n = 1 \text{ for some } n \leq 3, \\ t_\ell \neq 2 \text{ for some } \ell > n \text{ and} \\ t_\ell \neq 0 \text{ for some } \ell > n \end{array} \right\}.$$

   (Then $C_3 = [0, 1] \setminus G_3$ is the third set developed in the creation of C.) Sketch the graph of $\varphi|_{G_3}$, labeling all the values it takes. What can you infer about φ in general?

   (b) Show that φ extends, uniquely, to a continuous function from $[0, 1]$ to $[0, 1]$.

   [Again, this is a good time to pull out the PMath 351 toolbox.]

   We denote this extended function, again, by φ. It is called the *Cantor ternary function.*

   (c) Define $\psi : [0, 1] \to [0, 2]$ by $\psi(t) = \varphi(t) + t$. Show that ψ is strictly increasing and that $\psi(C)$ is a closed nowhere dense subset of $[0, 2]$, which has measure 1.

   (d) Find an example of a measurable function $f : \mathbb{R} \to \mathbb{R}$ and a continuous function $h : \mathbb{R} \to \mathbb{R}$ such that $f \circ h$ is *not* measurable.

   [Hint: Use q. 1 to aid in your choice of f, and (c) to aid in your choice of h.]

[*Don't forget the next page ...*]

3. Let $[a,b]$ be a bounded interval in $\mathbb{R}$ and $f:[a,b]\to\mathbb{R}$ be a bounded function. If $\mathcal{P}=\{a=x_0<x_1<\cdots<x_n=b\}$ is a partition of $[a,b]$ by intervals, then let

$$L(f,\mathcal{P})=\sum_{i=1}^n m_i(f,\mathcal{P})(x_i-x_{i-1}) \quad \text{and} \quad U(f,\mathcal{P})=\sum_{i=1}^n M_i(f,\mathcal{P})(x_i-x_{i-1})$$

where $m_i(f,\mathcal{P})=\inf_{x\in[x_{i-1},x_i]}f(x)$ and $M_i(f,\mathcal{P})=\sup_{x\in[x_{i-1},x_i]}f(x)$, for each i.

Note: These "upper" and "lower" sums make sense as $\mathbb{R}$ is totally ordered. If we replace the codomain $\mathbb{R}$ by a Banach space $\mathcal{X}$, we have no total ordering on $\mathcal{X}$ and cannot make sense of these at all.

(a) Show that f is Riemann integrable on $[a,b]$ if and only if for each $\varepsilon>0$, there is a partition $\mathcal{P}_\varepsilon$ such that

$$U(f,\mathcal{P})-L(f,\mathcal{P})<\varepsilon \quad \text{whenever} \quad \mathcal{P}\supset\mathcal{P}_\varepsilon.$$

[Verify Cauchy criterion for Riemann integrability.]

(b) For each n in $\mathbb{N}$ let

$$E_n=\left\{x\in[a,b]:\begin{array}{c}\text{for every }\delta>0,\text{ there exist }y,z\text{ in}\\(x-\delta,x+\delta)\cap[a,b]\text{ s.t. }|f(y)-f(z)|\geq\frac{1}{n}\end{array}\right\}.$$

Show that $E=\bigcup_{n=1}^\infty E_n$ is the set of points at which f is discontinuous.

(c) Show that if f is Riemann integrable, then $\lambda^*(E_n)=0$ for each n. Hence deduce that the set of points of discontinuity of f is null.

(d) Show that if f is Riemann integrable, then it is Lebesgue integrable, and the two integrals are equal.

We note that the converse to (c) holds too. Hence *a function $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuities has Lebesgue measure 0.*

4. Let $a<b$ in $\overline{\mathbb{R}}$ and $f:(a,b)\to\mathbb{R}$. We say that f is "improperly Riemann integrable" on (a,b) if

(i) for any $a<x<y<b$, f is Riemann integrable on $[x,y]$, and

(ii) $\lim_{x\to a^+}\lim_{y\to b^-}\int_x^y f$ exists.

In this case, the quantity in (ii) is called the *improper Riemann integral* of f on (a,b) and is denoted $\int_a^b f$.

(a) Show that if f is improperly Riemann integrable on (a,b), and $f\geq 0$, then f is Lebesgue integrable with $\int_a^b f=\int_{(a,b)}f$ [i.e. improper Riemann int. = Lebesgue int.].

(b) Does the conclusion of (a) hold without the assumption that $f\geq 0$? Prove or provide a counter-example.