Pure Math 450, Assignment 3: Solution Sketch

1. Let $a > 0$ be so $A \subset (-a, a)$. Define and equivalence relation on A by
$$x \sim y \quad \Leftrightarrow \quad x - y \in \mathbb{Q}.$$
Let $E \subset A$ be a set of representative, one from each equivalence class. If $\{q_n\}_{n=1}^{\infty} = (-2a, 2a) \cap \mathbb{Q}$, then it is easily verified that
$$A \subset \dot{\bigcup_{n=1}^{\infty}} (q_n + E) \subset (-3a, 3a).$$
It can be shown, just as in the proof in class, that E is not measurable; i.e. it is not a null set, nor a measurable non-null set.

2. **(a)** Draw a pretty picture with formula:
$$\varphi|_{G_3} = \frac{1}{8}\chi_{(1/27,2/27)} + \frac{1}{4}\chi_{(1/9,2/9)} + \frac{3}{8}\chi_{(7/27,8/27)} + \frac{1}{2}\chi_{(1/2,2/3)} + \frac{5}{8}\chi_{(19/27,20/27)} + \frac{3}{4}\chi_{(7/9,8/9)} + \frac{7}{8}\chi_{(25/27,26/27)}.$$
The graph should look like a staircase. It appears this function is non-decreasing, but constant on each interval of G.

   (b) *Solution #1.* We have that that φ is uniformly continuous on G: the definition with $\varepsilon > 1/2^n$ will be satisfied with $\delta = 1/3^n$. Indeed, if $x = 0.x_1x_2\cdots < y = 0.y_1y_2\ldots$ in G (ternary expansions) we let $k = \min\{i : x_i \neq y_i\}$. If $y - x < 1/3^n$ then either $k > n$; or $k \leq n$ with $y_k - x_k = 1$, $y_i = 0$ and $x_i = 2$ for $k < i \leq n$ (why?). Verify in the first case that $0 \leq \varphi(y) - \varphi(x) < \sum_{i=n+1}^{\infty} 1/2^i = 1/2^n$; while in the second $0 \leq \varphi(y) - \varphi(x) \leq 1/2^k - \sum_{i=k+1}^{n} 1/2^i = 1/2^n$.
   Since G is dense in $[0, 1]$, φ extends uniquely to a continuous function on $[0, 1]$, by a result from PM351. Call this extension φ.
   Solution #2. If $c \in C$, write $c = \sum_{k=1}^{\infty} \varepsilon_k/3^k$ (uniquely) where $\varepsilon_k \in \{0, 2\}$ for each k. Then verify that $\lim_{x \in G, x \to c} \varphi(x) = \sum_{k=1}^{\infty} \varepsilon_k/2^k$, which we define as $\varphi(c)$. Check this seperately at "endpoints": $c = 0.\varepsilon_1 \ldots \varepsilon_{n-1}1000\cdots = 0.\varepsilon_1 \ldots \varepsilon_{n-1}0222\ldots$, or $c = 0.\varepsilon_1 \ldots \varepsilon_{n-1}1222\cdots = 0.\varepsilon_1 \ldots \varepsilon_{n-1}2000\ldots$ ($\varepsilon_i \in \{0, 2\}$ for $1 \leq i \leq n-1$); and also at those points in C which admit only one ternary representation.
   Solution #2'. Define φ on C as above. Verify that $\varphi : [0, 1] = C \cup G \to [0, 1]$ is non-decreasing [ideas from (c), below, will suffice] and that $\varphi(C) = [0, 1]$ (use that every element in $[0, 1]$ can be represented in binary expansion). Hence φ is non-decreasing and surjective. Verify that this implies that φ is continuous.
   Solution #3. Define a continuous function $f_n : [0, 1] \to [0, 1]$ by $f_n|_{G_n} = \varphi|_{G_n}$ and so the graph of f_n on each interval in C_n is a straight line segement. Verify that $\|f_n - f_{n+1}\|_{\infty} \leq 1/2^n$. Hence it follows that $(f_n)_{n=1}^{\infty}$ is uniformly Cauchy, and thus converges to a continuous function f. Since $f|_G = \varphi$, we see that f is the desired extension and we may write $f = \varphi$ on $[0, 1]$.

(c) Since $x \mapsto x$ is strictly increasing, it suffices to verify that φ is non-decreasing. Verify directly from the fomula for φ that if $x < y$ in G, then $\varphi(x) < \varphi(y)$. Now if $x < y$ in $[0,1]$ find sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty} \subset G$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then $\varphi(x) = \lim_{n\to\infty}\varphi(x_n) \leq \lim_{n\to\infty}\varphi(y_n) = \varphi(y)$. [Alternatively, using #2, above, let $x = 0.x_1x_2\cdots < y = 0.y_1y_2\ldots$ (ternary representations) in $[0,1]$, and let $k = \min\{n : y_n > x_n\}$. If $x_n = 1 = y_n$ for some $n < k$, $\varphi(x) = \varphi(y)$ (why?); otherwise calculate $\varphi(x)$ and $\varphi(y)$ to see that $\varphi(x) < \varphi(y)$.]

Thus $\psi(x) = \varphi(x) + x$ is a bijective continuous function from the compact set $[0,1]$ onto $[0,2]$. By a result from PM351, ψ is a homeomorphism. Thus it follows that $\psi(G)$ is open and dense in $[0,2]$, and hence $\psi(C)$ is closed and nowhere dense. Moreover, just as we can compute that $\lambda(G) = 1$ (in Assignment #2), we can compute that $\lambda(\psi(G)) = 1$. Hence it follows $\lambda(\psi(C)) = 1$ too.

(d) Let $h : \mathbb{R} \to \mathbb{R}$ be any continuous function for which $h|_{[0,2]} = \psi^{-1}$. Let E be a non-measurable subset in $\psi(C)$ (whose existence is given by q. 1, above]. Then check that $f = \chi_{\psi^{-1}(E)}$ is measurable and $(f \circ h)^{-1}\big((1/2,\infty)\big) = E$, so $f \circ h$ is not measurbale.

3. **(a)** Suppose f is Riemann integrable, and $\varepsilon > 0$ be given. We have Cauchy Criterion for Riemann sums: There is a partition $\mathcal{P}$ for which $|S(f,\mathcal{Q}) - S(f,\mathcal{R})| < \varepsilon/3$ whenever $\mathcal{Q}, \mathcal{R} \supset \mathcal{P}_\varepsilon$. Find specific Riemann sums $S_u(f,\mathcal{P})$ and $S_\ell(f,\mathcal{P})$ for which $|U(f,\mathcal{P}) - S_u(f,\mathcal{P})| < \varepsilon/3$ and $|L(f,\mathcal{P}) - S_\ell(f,\mathcal{P})| < \varepsilon/3$ (why is this possible?). Then we have $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$.

To see the converse, verify that for any partitions $\mathcal{P}, \mathcal{Q}$ of $[a,b]$ we have

$$L(f,\mathcal{P}) \leq L(f,\mathcal{P}\cup\mathcal{Q}) \leq U(f,\mathcal{P}\cup\mathcal{Q}) \leq U(f,\mathcal{Q}); \text{ similarly } L(f,\mathcal{Q}) \leq U(f,\mathcal{P}).$$

Let $\varepsilon > 0$ be given. If there is a partition $\mathcal{P}$ for which $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon/2$, then for partitions $\mathcal{P}, \mathcal{Q}$ with $\mathcal{Q}, \mathcal{R} \supset \mathcal{P}$ we have for any Riemann sums

$$L(f,\mathcal{Q}) \leq S(f,\mathcal{Q}) \leq U(f,\mathcal{Q}) \text{ and } L(f,\mathcal{R}) \leq S(f,\mathcal{R}) \leq U(f,\mathcal{R}).$$

This, combined with the inequalities three lines above yeilds $|S(f,\mathcal{P}) - S(f,\mathcal{Q})| < \varepsilon$. Draw a diagram of a real line segment to see why.

(b) Verify that for each n, the definition of continuity fails for f at $x \in E_n$. Indeed, try this with $\varepsilon = 1/(2n)$. On the other hand, if f is not continuous at x, the definition of continuity fails for some $\varepsilon > 0$. Find $n > 1/\varepsilon$ and see that $x \in E_n$. It follows that E is exactly the set of points in $[a,b]$ where f fails to be continuous.

(c) Let n be fixed, and $\varepsilon > 0$. Let, from (a), $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_m = b\}$ be so $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon/n$. Let $I_i = (x_{i-1}, x_i)$ for $i = 1,\ldots,m$. Then verify that

$$\frac{1}{n}\sum_{I_i\cap E_n\neq\varnothing} \ell(I_i) \leq \sum_{I_i\cap E_n\neq\varnothing} (M_i(f,\mathcal{P}) - m_i(f,\mathcal{P}))(x_i - x_{i-1}) \leq U(f,\mathcal{P}) - L(f,\mathcal{P}) < \frac{\varepsilon}{n}$$

from which it follows that E_n can be covered by a finite set (of endpoints) and a family of intervals the sum of whose lengths is less than ε. It follows that $\lambda^*(E_n) < \varepsilon$. As ε is arbitrary each E_n is a null set; the countable union of null sets is null.

(d) Find a sequence of Riemann sums $\big(S(f,\mathcal{P}_n)\big)_{n=1}^{\infty}$ with partitions $\mathcal{P}_n = \{a = s_0 < s_1 < \cdots < s_{m(n)} = b\}$ such that $\lim_{n\to\infty} S(f,\mathcal{P}_n) = \int_a^b f$ (Riemann integral of f). Write

$$S(f,\mathcal{P}_n) = \sum_{k=1}^{m(n)} f(x_{n,k}^*)(x_{n,k} - x_{n,k-1}).$$

Define for each n the measurable simple function

$$f_n = \sum_{k=1}^{m(n)} f(x_{n,k}^*)\chi_{[x_{n,k-1},x_{n,k})}.$$

Verify that for each $x \in [a,b] \setminus E$ that $\lim_{n\to\infty} f_n(x) = f(x)$. (This requires (c), above.) Then, if $M > 0$ is such that $\sup_{x\in[a,b]} |f(x)| \leq M$, we have that $|f_n| \leq M$ too. Thus the L.D.C.T. is satisfied, with the constant function M serving as an integrable majorant. Hence we get that f is Lebesgue integrable with Lebesgue integral

$$\int_{[a,b]} f = \lim_{n\to\infty} \int_{[a,b]} f_n = \lim_{n\to\infty} S(f,\mathcal{P}_n) = \int_a^b f.$$

Note: We can equally replace f by $f + M$. Then we can use $L(f,\mathcal{P}_n)$'s in place of $S(f,\mathcal{P}_n)$'s above and their corresponding simple functions. We should justify that $\lim_{n\to\infty} L(f,\mathcal{P}_n) = \int_a^b f$. Then use M.C.T. in place of L.D.C.T.

4. **(a)** Choose sequences $a < \cdots < a_2 < a_1 < b_1 < b_2 < \ldots b$ so $a = \lim_{n\to\infty} a_n$ and $b = \lim_{n\to\infty} b_n$. Let $f_n = f\chi_{[a_n,b_n]}$. Then $f_1 \leq f_2 \leq \ldots$ on (a,b) and $f = \lim_{n\to\infty} f_n$. We have using the definition of improper Rieman integration, then q. 3, a proposition we proved before M.C.T., then M.C.T. that

$$\int_a^b f = \lim_{n\to\infty} \int_{a_n}^{b_n} f = \lim_{n\to\infty} \int_{[a_n,b_n]} f = \lim_{n\to\infty} \int_{(a,b)} f_n = \int_{(a,b)} f.$$

In particular f is the limit of integrable, hence measurable functions, hence measurable; and $\int_{(a,b)} f = \int_a^b f < \infty$; so f is integrable.

(b) It is possible to be improperly Lebesgue integrable with $\int_{(a,b)} |f| = \infty$. For example let

$$f = \sum_{n=1}^{\infty} (-1)^n n\chi_{[1/(n+1),1/n)} \text{ on } (0,1)$$

Observe that

$$\int_x^1 f = (-1)^{\lceil 1/x\rceil -1}(\lceil 1/x\rceil - 1)\left(\frac{1}{\lceil 1/x\rceil - 1} - x\right) + \sum_{n=1}^{\lceil 1/x\rceil -1} (-1)^n \frac{1}{n+1}$$

which converges as $x \to 0^+$ (why?). However,

$$\int_x^1 |f| = (\lceil 1/x\rceil - 1)\left(\frac{1}{\lceil 1/x\rceil - 1} - x\right) + \sum_{n=1}^{\lceil 1/x\rceil -1} \frac{1}{n+1}$$

which grows without bound as $x \to 0^+$. It follows by procedure of (a), above that $\int_{(0,1)} |f| = \infty$.

Another example: $f(x) = \dfrac{\sin x}{x}$ on $(0,\infty)$, or $g(x) = \dfrac{\sin(1/x)}{x}$ on $(0,1)$. These require careful estimates to verify improper Riemann integrability, but failure of Lebesgue integrability.