Pure Math 450, Assignment 4: Solution Sketch

1. **(a)** First suppose $f \geq 0$ a.e. By a lemma, following M.C.T., we have a sequence of simple functions with $\varphi_n \leq f$ and $\lim_{n\to\infty} \varphi_n = f$ a.e. Thus $\lim_{n\to\infty} |\varphi_n - f|^p = 0$ a.e. and $|\varphi_n - f|^p \leq 2^p |f|^p$ a.e. (Why?) Then by L.D.C.T.

$$\|\varphi_n - f\|_p = \left(\int_a^b |\varphi_n - f|^p\right)^{1/p} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus for some n, $\|\varphi_n - f\|_p < \varepsilon$. For general f write $f = f^+ - f^-$ a.e.; then deal with positive parts individually and appeal to Minkowski's inequality.

(b) Consider first $\varphi = \chi_E$, where $E \subset [a, b]$ is measurable. Find a cover of E by open intervals $\{I_n\}_{n=1}^{\infty}$ so $\lambda(E) + (\varepsilon/2)^p > \sum_{n=1}^{\infty} \lambda(I_n)$, then find m so $\sum_{n=m+1}^{\infty} \lambda(I_n) < (\varepsilon/2)^p$. Let $J_1 = I_1 \cap [a, b]$, $J_2 = (I_2 \setminus I_1) \cap [a, b], \ldots J_m = (I_m \setminus \bigcup_{n=1}^{n-1} I_n) \cap [a, b]$, so $J_i \cap J_j = \varnothing$ for $i \neq j$ and $\bigcup_{n=1}^{n} J_n = \bigcup_{n=1}^{n} I_n$. Now

$$\lambda\left(E \setminus \bigcup_{n=1}^{m} J_n\right) \leq \lambda\left(\bigcup_{n=m+1}^{\infty} J_n\right) < (\varepsilon/2)^p$$

(why?), and

$$\sum_{n=1}^{m} \lambda(J_n \setminus E) = \lambda\left(\bigcup_{n=1}^{m} I_n \setminus E\right) \leq \lambda\left(\bigcup_{n=1}^{\infty} I_n \setminus E\right) < (\varepsilon/2)^p$$

(why?). Then verify that

$$\begin{aligned}
\left\|\sum_{n=1}^{m} \chi_{J_n} - \chi_E\right\|_p &= \left\|\sum_{n=1}^{m} (\chi_{J_n} - \chi_{E\cap J_n}) - \chi_{E\setminus \bigcup_{n=1}^{m} J_n}\right\|_p \\
&= \left\|\sum_{n=1}^{m} \chi_{J_n\setminus E}\right\|_p + \left\|\chi_{E\setminus \bigcup_{n=1}^{m} J_n}\right\|_p \\
&= \left(\int_a^b \left|\sum_{n=1}^{m} \chi_{J_n\setminus E}\right|^p\right)^{1/p} + \left(\int_a^b |\chi_{E\setminus \bigcup_{n=1}^{m} J_n}|^p\right)^{1/p} \\
&= \left(\sum_{n=1}^{m} \lambda(J_n \setminus E)\right)^{1/p} + \lambda\left(E \setminus \bigcup_{n=1}^{m} J_n\right)^{1/p} < \varepsilon.
\end{aligned}$$

Now if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$, with $a_i \neq 0$ for any i, we can find step function ψ_i so $\|\psi_i - \chi_{E_i}\|_p < \varepsilon/|a_i|$. Then $\psi = \psi_1 + \cdots + \psi_n$ is a step function with $\|\psi - \varphi\|_p < \varepsilon$.

(c) Write $\psi = \sum_{j=1}^{m} b_j \chi_{I_j}$ where $b_j \neq 0$ for each j. For each j find $h_j \in C[a, b]$ such that $0 \leq h_j \leq 1$, $\mathrm{supp}(h_j) \subset \overline{I_j} = [c_j, d_j]$ and $h_j(t) = 1$ for t in $[c_j + (\varepsilon/|b_j|)^p/2, d_j -$

$(\varepsilon/|b_j|)^p/2]$. (Each h_j can be chosen to be piecewise affine; draw a picture to see why.) Then

$$\left\|h_j - \chi_{I_j}\right\|_p \leq \frac{\varepsilon}{|b_j|}$$

(Draw a picture to see how to make this estimate.) Let $h = b_1h_1 + \cdots + b_nh_n$ and an application of Minkowski's inequality establishes $\|h - \psi\|_p < \varepsilon$.

Supplemental: on a.e. convergence. Suppose that $E \subset [a,b]$ is measurable. We find, as in the solution to A2 Q1(b), open $G_n = \dot{\bigcup}_{j=1}^{\infty}(c_{j,n}, d_{j,n})$ so $E \subset G_n$ and $\lambda(G_n) < \lambda(E)+\frac{1}{n}$; we may also suppose $G_n \supset G_{n+1}$ (why?). Recall from A2 Q2(b) that $A = \bigcap_{n=1}^{\infty} G_n$ satisfies $\lambda(A\backslash E) = 0$ so $\chi_A = \chi_E$ a.e. Let $\psi_n = \sum_{j=1}^{n} \chi_{(c_{j,n},d_{j,n})}$. Then $\lim_{n\to\infty} \psi_n = \chi_A$ pointwise (why?). Now if h_n is built as above, for choice $\varepsilon = \frac{\ell_n}{n}$, where $\ell_n = \min\{c_{j,n} - d_{j,n} : j = 1, \ldots, n \text{ and } (c_{j,n}, d_{j,n}) \neq \varnothing\}$, $b_j = 1$ and $p = 1$. We find that $\|h_n\|_\infty \leq 1$ and $\lim_{n\to\infty} h_n = \lim_{n\to\infty} \psi_n = \chi_A$ pointwise (why?). Hence $\lim_{n\to\infty} h_n = \chi_E$ a.e.

Now suppose that $\varphi = \chi_{E_1} - \chi_{E_2}$ where E_1 and E_2 are measurable with $E_1 \cap E_2 = \varnothing$. Find for $k = 1, 2$ a sequence of continuous functions $(h_{k,n})_{n=1}^{\infty}$ so $\|h_{k,n}\|_\infty \leq 1$ and $\lim_{n\to\infty} h_{k,n} = \chi_{E_k}$ a.e. Then $h_n = h_{1,n} - h_{2,n}$ defines a sequence of continuous functions satisfying $\|h_n\|_\infty \leq 1$ and $\lim_{n\to\infty} h_n = \varphi$ a.e.

This is used with $\varphi = \mathrm{sgn}\circ f$ in the L_1-functional Theorem, in class.

(d) If $h \in C[a,b]$ we have

$$\|h\|_p{}^p = \int_a^b |h(t)|^p dt \leq \int_a^b \|h\|_u{}^p dt = \|h\|_u^p (b-a)$$

so $\|h\|_p \leq (b-a)^{1/p} \|h\|_u$. [An identical argument shows $L_\infty[a,b] \subset L_p[a,b]$, with the same norm inequality.]

If $h \neq 0$, let t_0 be so $|h(t_0)| > 0$ and find $\delta > 0$ be so $|t - t_0| < \delta$ implies $|h(t)| > |h(t_0)|/2$. W.l.o.g. we can assume $t_0 \in (a,b)$ and $(t_0 - \delta, t_0 + \delta) \subset [a,b]$ (why?). Then

$$\|h\|_p{}^p = \int_a^b |h|^p \geq \int_{t_0-\delta}^{t_0+\delta} |h|^p \geq \delta|h(t_0)| > 0$$

so $\|h\|_p \neq 0$.

(e) This is a standard 3ε-argument, using Minkowski's inequality.

(f) It is trivial that $\|h\|_u$ is an essential bound for h, and thus $\|h\|_u \geq \operatorname{ess\,sup}_{x\in[a,b]} |h(x)| = \|h\|_\infty$. Conversely, there is x_0 so $|h(x_0)| = \|h\|_u$ (why?). Given $\varepsilon > 0$, find $\delta > 0$ so $|x - x_0| < \delta$ implies $|h(x)| > \|h\|_u - \varepsilon$. Then $(x_0 - \delta, x_0 + \delta) \cap [a,b]$ is a non-null set on which $|h(x)| > \|h\|_u - \varepsilon$. It follows that $\|h\|_\infty \geq \|h\|_u - \varepsilon$.

2. **(a)** If $f \in L_p(\mathbb{R})$, then $\lim_{n\to\infty} |f - f\chi_{[-n,n]}| = 0$ a.e. and $|f - f\chi_{[-n,n]}| \leq 2^p|f|^p$ a.e. Then using L.D.C.T. $\lim_{n\to\infty} \left\|f - f\chi_{[-n,n]}\right\|_p = 0$, so there is n so $\left\|f - f\chi_{[-n,n]}\right\|_p < \varepsilon/3$. Using part (e) of the question above, there is $g \in C[-n,n]$ such that

$\left(\int_{-n}^{n} |g-f|^p\right)^{1/p} < \varepsilon/3$. Finally, we can find $h \in C_c(\mathbb{R})$ so $h|_{[-n,n]} = g$ and $\mathrm{supp}(h) \subset [-n-(\varepsilon/3(|g(-n)|+1)^p, n+(\varepsilon/3(|g(n)|+1)^p]$, suitably bounded so $\|h-g\|_p < \varepsilon/3$. (Draw pictures to see why this proof works.)

(b) Let $f \in C_c(\mathbb{R})$, $\varepsilon > 0$. Find N so $|x| > N$ implies $|f(x)| < \varepsilon/2$. Find $h \in C_c(\mathbb{R})$ so $h|_{-N,N]} = f|_{-N,N]}$ and $|h(x)| < \varepsilon/2$ for $|x| > N$. (We can find h so $\mathrm{supp}(h) \subset [-N-\delta, N+\delta]$ for any $\delta > 0$.) Then

$$\|f-h\|_\infty = \sup_{|x|>N} |f(x)-h(x)| \leq \sup_{|x|>N} |f(x)| + \sup_{|x|>N} |h(x)| < \varepsilon.$$

(c) This is false for $1 \leq p < \infty$. Consider

$$f(x) = \begin{cases} 1 & \text{for } -1 \leq x \leq 1 \\ \frac{1}{|x|^{1/p}} & \text{for } |x| > 1. \end{cases}$$

For $p = \infty$, $C_0(\mathbb{R})$ is a closed subspace of $L_\infty(\mathbb{R})$. The proof is nearly identical to that of 2 (f).

3. **(a)** Given $\varepsilon > 0$, we choose, first, $1 > \delta > 0$ so $|t-0| < \delta$ implies $|\varphi(t)-\varphi(0)| < \varepsilon/4$, and then choose N so $n \geq N$ implies

$$\int_{[-1,-\delta]} f_n + \int_{[\delta,1]} f_n < \frac{\varepsilon}{6\,\|\varphi\|_\infty} \quad \text{and} \quad \left|\int_{[-1,1]} f_n - 1\right| < \min\left(\frac{\varepsilon}{3\,\|\varphi\|_\infty}, \frac{1}{3}\right)$$

where we may assume $\|\varphi\|_\infty > 0$. Thus for $n \geq N$ we can (and must) check that

$$\begin{aligned} \left|\varphi(0) - \int_{[-1,1]} \varphi f_n\right| &\leq \left|\varphi(0) - \varphi(0)\int_{[-1,1]} f_n\right| + \left|\int_{[-1,1]} [\varphi(0)-\varphi] f_n\right| \\ &< \frac{\varepsilon}{3} + 2\,\|\varphi\|_\infty \int_{[-1,-\delta]\cup[\delta,1]} f_n + \int_{(-\delta,\delta)} |\varphi(0)-\varphi| f_n \\ &\leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{4}\int_{[-1,1]} f_n < \varepsilon. \end{aligned}$$

(b) Consider either of the sequences given by

$$f_n = n\chi_{(0,1/n]} \text{ or } f_n = \frac{n}{2}\chi_{[-1/n,1/n]}.$$

Note that the first sequence converges pointwise to 0 everywhere, whilst the second does so on $[-1,1] \setminus \{0\}$.