Pure Math 450, Assignment 5

Due: March 16.

Notation: Since Lebesgue integration extends Riemann integration, we will often use notation $\int_a^b f = \int_{[a,b]} f = \int_{(a,b)} f$ interchangably. We make an exception in Q. 2 (c), below.

1. (Translation and inversion invariance of Lebesgue integration.) Let $f \in L(\mathbb{R})$ (integrable function space). Verify that and $t \in \mathbb{R}$.

   (a) $t{*}f \in L(\mathbb{R})$ where $t{*}f(x) = f(x-t)$, and

   (b) $\check{f} \in L(\mathbb{R})$ where $\check{f}(x) = f(-x)$,

   with
$$\int_{\mathbb{R}} t{*}f = \int_{\mathbb{R}} f = \int_{\mathbb{R}} \check{f}.$$
   Thus deduce that if $f \in L(\mathbb{T})$ (a.e. 2π-periodic functions, integrable on $[-\pi, \pi]$) and $t \in \mathbb{R}$ then

   (c) $t{*}f \in L(\mathbb{T})$ and $\check{f} \in L(\mathbb{T})$

   with
$$\int_{-\pi}^{\pi} t{*}f = \int_{-\pi}^{\pi} f = \int_{-\pi}^{\pi} \check{f}$$
   [Hint: For (a), (b) first consider simple functions, then non-negative functions, etc.]

2. (A concrete approach to convolutions.) Let $f \in C(\mathbb{T})$ and $g \in L(\mathbb{T})$. Assume, for simplicity, that each of f and g are real-valued.

   (a) Show that for each t in $[-\pi, \pi]$, $s \mapsto f(s)g(t-s)$ is integrable on $[-\pi, \pi]$.

   We then define $f{*}g : \mathbb{R} \to \mathbb{R}$ by
$$f{*}g(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)g(t-s)ds \overset{\text{(TI)}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t+s)g(-s)ds \overset{\text{(II)}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t-s)g(s)ds$$
   where (TI) holds by translation invariance, and (II) holds by inversion invariance.

   (b) Show that $f{*}g$ is continuous and 2π-periodic, i.e. $f{*}g \in C(\mathbb{T})$. Also, show that if $g_1, g_2 \in L(\mathbb{T})$ with $g_1 = g_2$ a.e. then $f{*}g_1 = f{*}g_2$.

   (c) Show that the following "Fubini Theorem" holds in this case:
   For any fixed s, the function $t \mapsto f(t-s)g(s)$ is Riemann integrable and
$$\int_{[-\pi,\pi]} \left(\int_{-\pi}^{\pi} f(t-s)g(s)dt \right) ds = \int_{-\pi}^{\pi} \left(\int_{[-\pi,\pi]} f(t-s)g(s)ds \right) dt$$
   where $\int_{-\pi}^{\pi} \cdots dt$ denotes Riemann integration while $\int_{[-\pi,\pi]} \cdots ds$ denotes Lebesgue integration.

(d) Deduce that $\|f*g\|_1 \le \|f\|_1 \|g\|_1$.

(e) Show that if g is continuous, then $\|f*g\|_\infty \le \|f\|_1 \|g\|_\infty$.

3. (Naive convergence of Fourier series fails catastrophically.)

(a) Fix t_0 in $[-\pi, \pi]$. Show that there is a subset U of $C(\mathbb{T})$, whose complement is meager (first category), such that

$$\sup_{n\in\mathbb{N}} |s_n(h, t_0)| = +\infty$$

if $h \in U$.

[Hint: Show that the family of linear functionals $\Gamma_n : C(\mathbb{T}) \to \mathbb{C}$, given for each n by $\Gamma_n(h) = s_n(h, t_0)$, is unbounded.]

(b) If $(t_m)_{m\in\mathbb{N}}$ is any sequence in $[-\pi, \pi]$, show that there exists a function h in $C(\mathbb{T})$ such that

$$\sup_{n\in\mathbb{N}} |s_n(f, t_m)| = +\infty$$

for each m in $\mathbb{N}$. Hence deduce that there is an h in $C(\mathbb{T})$ such that $\lim_{n\to\infty} s_n(h) \ne h$ (pointwise) on a dense subset of $[-\pi, \pi]$.

[Hint: If $U_1, U_2, \dots$ is a sequence of sets whose complements are each meager, then $\bigcap_{n=1}^\infty U_n$ enjoys this property too.]

4. Let K_n denote the Fejer kernel of order n.

(A special case of Fejer's Theorem.) Show that if $f \in L(\mathbb{T})$ and $x \in [-\pi, \pi]$ and we have that $\omega_f(x) = +\infty$, then

$$\lim_{n\to\infty} \sigma_n(f, x) = +\infty.$$

[Hint: Given any $M > 0$, show that there is $\delta > 0$ such that $\lim_{n\to\infty} \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)\big(f(x-s) + f(x+s)\big)ds \ge M$, while $\lim_{n\to\infty} \left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_n(s)\big(f(x-s) + f(x+s)\big)ds = 0$.]