Pure Math 450, Assignment 5: Solution Sketch

1. **(a)** We saw that $\lambda^*(t+E) = \lambda^*(E)$ for every $E \subset \mathbb{R}$, and E is measurable iff $t+E$ is measurable; hence $\lambda(E) = \lambda(t+E)$ for measurable E.

   We can show for $f : \mathbb{R} \to \mathbb{R}$ and $\alpha \in \mathbb{R}$ that

   $$(t{*}f)^{-1}\big((\alpha,\infty)\big) = t + \big[f^{-1}\big((\alpha,\infty)\big)\big]$$

   so $f \in \mathcal{M}(\mathbb{R})$ iff $t{*}f \in \mathcal{M}(\mathbb{R})$.

   If $\lambda(E) < \infty$ then $t{*}\chi_E = \chi_{t+E}$ (check this!) so $\int_{\mathbb{R}} \chi_{t+E} = \lambda(t+E) = \lambda(E) = \int_{\mathbb{R}} \chi_E$. Thus the same holds for any simple function. If $f \in \mathcal{M}^+(\mathbb{R})$ then we can check that

   $$\begin{aligned}\int_{\mathbb{R}} t{*}f &= \sup\left\{\int_{\mathbb{R}} \varphi : \varphi \in \mathcal{S}^+_{t*f}(\mathbb{R})\right\}\\ &= \sup\left\{\int_{\mathbb{R}} t{*}\varphi : \varphi \in \mathcal{S}^+_{f}(\mathbb{R})\right\},\ \text{i.e. } \mathcal{S}^+_{t*f}(\mathbb{R}) = t{*}\mathcal{S}^+_{f}(\mathbb{R})\\ &= \sup\left\{\int_{\mathbb{R}} \varphi : \varphi \in \mathcal{S}^+_{f}(\mathbb{R})\right\} = \int_{\mathbb{R}} f.\end{aligned}$$

   Now $f \in L(\mathbb{R})$ iff $f^+, f^- \in \mathcal{M}(\mathbb{R})$ with $\int_{\mathbb{R}} f^+, \int_{\mathbb{R}} f^- < \infty$. Thus if $f \in L(\mathbb{R})$ then $(t{*}f)^+ = t{*}f^+, (t{*}f)^- = t{*}f^- \in \mathcal{M}(\mathbb{R})$ with $\int_{\mathbb{R}} t{*}f^+ = \int_{\mathbb{R}} f^+, \int_{\mathbb{R}} t{*}f^- = \int_{\mathbb{R}} f^- < \infty$. Thus $t{*}f \in L(\mathbb{R})$ with $\int_{\mathbb{R}} t{*}f = \int_{\mathbb{R}} f$.

   Alternate. Use a corollary we gave to MCT to find a sequence of measurable simple functions $(\varphi_n)_{n=1}^\infty$ for which $\lim_{n\to\infty} \varphi_n = f$ (pointwise) and $\varphi_1^+ \leq \varphi_2^+ \leq \cdots \leq f^+$, and similar for f^-. For each n we write φ_n in standard form — $\varphi_n^+ = \sum_{j=1}^{m_n} a_{n,j}\chi_{E^+_{n,j}}$ — and we have

   $$t * \varphi_n^+ = \sum_{j=1}^{m_n} a_{n,j}\chi_{t+E^+_{n,j}}.$$

   Thus, as above, we see that $\int_{\mathbb{R}} t * \varphi_n^+ = \int_{\mathbb{R}} \varphi_n^+$. It follows $t * f^+ = t * (f^+)$, being pointwise limit of $(t * \varphi_n^+)_{n=1}^\infty$, is measurable and by MCT $\int_{\mathbb{R}} t * f^+ = \lim_{n\to\infty} \int_{\mathbb{R}} t * \varphi_n^+ = \lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n^+ = \int_{\mathbb{R}} t * f^+ < \infty$. Doing the same to f^- we see that $t{*}f$ is integrable with $\int_{\mathbb{R}} t * f = \int_{\mathbb{R}} f$.

   (b) First, use the definition of λ^* to show $\lambda^*(-E) = \lambda^*(E)$ for each $E \in \mathbb{R}$. Then one must verify that $E \in \mathcal{L}(\mathbb{R})$ iff $-E \in \mathcal{L}(\mathbb{R})$. Indeed, if $A \subset \mathbb{R}$ then

   $$\begin{aligned}\lambda^*(A \cap (-E)) + \lambda^*(A \cap (\mathbb{R} \setminus (-E))) &= \lambda^*(-[(-A) \cap E]) + \lambda^*(-[(-A) \cap (\mathbb{R} \setminus E)])\\ &= \lambda^*((-A) \cap E) + \lambda^*((-A) \cap (\mathbb{R} \setminus E)) = \lambda^*(-A) = \lambda^*(A).\end{aligned}$$

   Thus for measurable E we have $\lambda(-E) = \lambda(E)$.

From this point, the demonstration above can be repeated nearly verbatim, with $\check{f}$ substituted for $t{*}f$. Note that

$$(\check{f})^{-1}\big((\alpha,\infty)\big) = -f^{-1}\big((\alpha,\infty)\big)$$

and $\check{\chi}_E = \chi_{-E}$.

(b) Now if $f \in L(\mathbb{T})$ then $f\chi_{[-\pi,\pi]} \in L[-\pi,\pi]$ wirh $\int_{\mathbb{R}} f\chi_{[-\pi,\pi]} = \int_{-\pi}^{\pi} f$. [We saw this in class for $f \geq 0$, otherwise write $f = f^+ - f^-$.] Clearly $t{*}f$ is measurable and a.e. 2π-periodic. Write $t = 2\pi n + t'$ where $t' \in [-\pi,\pi)$ and we can verify, by 2π-peridicity and similar manipulations as above, that

$$\int_{-\pi}^{\pi} t{*}f = \int_{\mathbb{R}} t'{*}f\chi_{[-\pi,\pi]} = \int_{\mathbb{R}} f\chi_{[-\pi-t',\pi-t']} = \int_{-\pi-t'}^{\pi-t'} f.$$

If $t' \geq 0$ we have

$$\int_{-\pi-t'}^{\pi-t'} f = \int_{-\pi-t'}^{-\pi} f + \int_{-\pi}^{\pi-t'} f = \int_{\mathbb{R}} f\chi_{[-\pi-t',-\pi]} + \int_{-\pi}^{\pi-t'} f$$
$$= \int_{\mathbb{R}} (2\pi) * \big(f\chi_{[-\pi-t',-\pi]}\big) + \int_{-\pi}^{\pi-t'} f = \int_{\mathbb{R}} f\chi_{[\pi-t',\pi]} + \int_{-\pi}^{\pi-t'} f = \int_{-\pi}^{\pi} f$$

and a parallel proof holds if $t' \leq 0$.

Similarly

$$\int_{-\pi}^{\pi} \check{f} = \int_{\mathbb{R}} \check{f}\chi_{[-\pi,\pi]} = \int_{\mathbb{R}} f\check{\chi}_{[-\pi,\pi]} = \int_{-\pi}^{\pi} f.$$

2. **(a)** f measurable and bounded, g integrable. Thus $f\, t{*}\check{g}$ integrable.

(b) By uniform continuity of f, given $\varepsilon > 0$ there is $\delta > 0$ so $|t - t_0| < \delta$ implies $\|t{*}f - t_0{*}f\|_\infty < \varepsilon/2\pi(\|g\|_1 + 1)$. Then verify that

$$|f{*}g(t) - f{*}g(t_0)| \leq \int_{-\pi}^{\pi} |f(t-s) - f(t_0 - s)||g(s)|ds \leq \left\|t{*}\check{f} - t_0{*}\check{f}\right\|_\infty \|g\|_1 < \varepsilon.$$

Also if $g_1 = g_2$ a.e. we have for all t

$$f{*}g_1(t) - f{*}g_2(t) = \int_{-\pi}^{\pi} f(t-s)\big(g_1(s) - g_2(s)\big)ds = 0.$$

(c) $t \mapsto f(t-s)g(s)$ is continuous, and hence Riemann integrable. Let for each n, $\mathcal{P}_n = \{-\pi = t_{n,0} < t_{n,1} < \cdots < t_{n,m(n)} = \pi\}$ be a partition of $[-\pi,\pi]$; and let these be chosen so $\lim_{n\to\infty} L(\mathcal{P}_n) = 0$. Then by theory of the Riemann integral (on a continuous function)

$$\int_{-\pi}^{\pi} f(t-s)g(s)dt = \lim_{n\to\infty} \sum_{i=1}^{m(n)} f(t_{n,i} - s)g(s)(t_{n,i} - t_{n,i-1})$$

for each s. Verify that each of the fuctions $s \mapsto \sum_{i=1}^{m(n)} f(t_{n,i}-s)g(s)(t_{n,i}-t_{n,i-1})$ is majorised by $2\pi \|f\|_\infty |g|$, which is integrable by assumption. Hence by L.D.C.T. we have

$$\begin{aligned}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(t-s)g(s)dt\right)ds &= \lim_{n\to\infty}\int_{-\pi}^{\pi}\sum_{i=1}^{m(n)} f(t_{n,i}-s)g(s)(t_{n,i}-t_{n,i-1})ds \\ &= \lim_{n\to\infty}\sum_{i=1}^{m(n)}\left(\int_{-\pi}^{\pi} f(t_{n,i}-s)g(s)ds\right)(t_{n,i}-t_{n,i-1}) \\ &= \lim_{n\to\infty}\sum_{i=1}^{m(n)} f{*}g(t_{n,i})(t_{n,i}-t_{n,i-1}) \\ &= \int_{-\pi}^{\pi} f{*}g(t)dt = \int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(t-s)g(s)ds\right)dt\end{aligned}$$

where the second last inequalty holds by the fact $f{*}g$ is continuous and hence Riemann integrable.

(d) Applying the result above to $|f|$ and $|g|$ we obtain

$$\begin{aligned}\|f{*}g\|_1 &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)||g(s)|ds\right)dt = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi}|f(t-s)||g(s)|dt\right)ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)|dt\right)}_{=\|f\|_1,\ \text{by (TI), (II)}}|g(s)|ds = \|f\|_1\|g\|_1.\end{aligned}$$

(e) For any t we have

$$|f{*}g(t)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(s-t)||g(s)|ds \leq \int_{-\pi}^{\pi}|f(s-t)|\,\|g\|_\infty\,ds = \|f\|_1\|g\|_\infty$$

where we use translation invariance (TI) and inversion invariance (II).

3. **(a)** We have, using ideas from q. 2,

$$\begin{aligned}s_n(h,t_0) = D_n{*}f(t_0) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(s)f(t_0-s)ds = \int_{-\pi}^{\pi} D_n(t_0+s)f(-s)ds \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\check{D}_n(s-t_0)f(s)ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} t_0{*}D_n(s)f(s)ds,\ \text{since } D_n = \check{D}_n \\ &= \Gamma_{t_0*D_n}(f)\end{aligned}$$

where $\Gamma_{t_0*D_n} : C(\mathbb{T}) \to \mathbb{C}$ is the usual functional with norm $\|\Gamma_{t_0*D_n}\|_* = \|t_0{*}D_n\|_1 = \|D_n\|_1 = L_n$, where the Lebesgue constants $L_n \overset{n\to\infty}{\longrightarrow} \infty$. By a corollary to Banach-Stienhaus, there is a set $U \subset C(\mathbb{T})$, whose complement is meager, so for $h \in U$

$$\lim_{n\to\infty}|s_n(f,t_0)| = \lim_{n\to\infty}|\Gamma_{t_0*D_n}(f)| = \infty.$$

(b) For each m let U_m be a set of meager complement such that for $h \in U_m$, $\sup_{n\in\mathbb{N}} |s_n(h,t_m)| = \infty$. Write $U_m = \mathbb{R} \setminus F_m$ where F_m is meager. Then $F = \bigcup_{m=1}^{\infty} F_m$ is meager (why?) so $\bigcap_{m=1}^{\infty} U_m = \mathbb{R} \setminus F$ has meager complement. It is clear that if $h \in U$ then $\sup_{n\in\mathbb{N}} |s_n(h,t_m)| = \infty$.

We can find $\{t_m\}_{m=1}^{\infty}$ dense in $[-\pi,\pi]$; say an enumeration of $\mathbb{Q} \cap [-\pi,\pi]$.

4. Let $M > 0$ and find $\delta > 0$ so $0 < |s| < \delta$ implies $\frac{1}{2}\big(f(x-s)+f(x+s)\big) \geq M$. Then for each n

$$\begin{aligned}\sigma_n(f,x) = K_n * f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(s)f(x-s)ds \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)f(x-s)ds + \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_n(s)f(x-s)ds \\ &= \frac{1}{4\pi}\int_{-\delta}^{\delta} K_n(s)\big(f(x+s)+f(x-s)\big)ds + \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right) K_n(s)f(x-s)ds\end{aligned}$$

Verify that

$$\begin{aligned}\left|\int_{-\pi}^{-\delta} K_n(s)f(x-s)ds\right| &\leq \frac{\pi^2}{4(n+1)\delta^2}\int_{-\pi}^{-\delta} |x * \check{f}| \\ &\leq \frac{\pi}{4(n+1)\delta^2}\|f\|_1 \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Similarly $\int_{\delta}^{\pi} K_n(s)f(x-s)ds \overset{n\to\infty}{\longrightarrow} 0$. We have, by choice of δ, above, that

$$\begin{aligned}&\liminf_{n\to\infty} \frac{1}{4\pi}\int_{-\delta}^{\delta} K_n(s)\big(f(x+s)+f(x-s)\big)ds \\ &\qquad \geq \liminf_{n\to\infty} \frac{1}{2\pi}\int_{-\delta}^{\delta} K_n(s)M ds \\ &\qquad = \liminf_{n\to\infty} \frac{M}{2\pi}\left(\int_{-\pi}^{\pi} K_n(s)ds - \left[\int_{-\pi}^{-\delta} + \int_{\delta}^{\pi}\right] K_n(s)ds\right) \\ &\qquad = \lim_{n\to\infty} \frac{M}{2\pi}\left(2\pi - \int_{-\pi}^{-\delta} K_n - \int_{\delta}^{\pi} K_n\right) = M\end{aligned}$$

Putting it all together we find

$$\liminf_{n\to\infty} \sigma_n(f,x) \geq M$$

Since $M > 0$ is arbitrary, we see that $\lim_{n\to\infty} \sigma_n(f,x) = \infty$. (Is this obvious?)