Pure Math 450, Assignment 6

Due: Friday, March 30.

1. **(a)** Show that if $f \in L(\mathbb{T})$ and is *a.e. even*, i.e. $f(-t) = f(t)$ for a.e. t in $\mathbb{R}$, then f has Fourier sums
$$s_n(f,t) = c_0(f) + \sum_{k=1}^{n} 2c_k(f)\cos kt$$
and $c_k(f) = c_{-k}(f) = \frac{1}{\pi}\int_0^\pi f(s)\cos ks\, ds$ for each k in $\mathbb{Z}$.

   (b) Let $f(t) = \chi_{[-\pi/2,\pi/2]}(t)$ if $t \in [-\pi,\pi]$, and extend f 2π-periodically to all of $\mathbb{R}$. Compute $c_k(f)$ for $k = 0, 1, 2, \ldots$.

   (c) Use results in (b) to evaluate each of the series $\sum_{j=0}^{\infty}\frac{(-1)^j}{2j+1}$ (no surprise), $\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2}$. Indicate any major theorems which are used to justify your computations.

   (d) Let $\alpha > 0$ and $g(t) = \cosh(\alpha t) = \frac{1}{2}(e^{\alpha t}+e^{-\alpha t})$ if $t \in [-\pi,\pi]$, and extend g 2π-periodically to all of $\mathbb{R}$. Compute $c_k(f)$ for $k = 0, 1, 2, \ldots$

   (e) Evaluate each of the series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2+\alpha^2}$, $\sum_{k=1}^{\infty}\frac{1}{k^2+\alpha^2}$ and $\sum_{k=1}^{\infty}\frac{1}{(k^2+\alpha^2)^2}$. Indicate any major theorems which are used to justify your computations.

2. (The *Fourier algebra.*) Let
$$A(\mathbb{T}) = \left\{ f \in L_1(\mathbb{T}) : \sum_{n=-\infty}^{\infty} |c_n(f)| < +\infty \right\}.$$

   (a) If f in $A(\mathbb{T})$ show that $\big(s_n(f)\big)_{n=1}^{\infty}$ is a uniformly Cauchy sequence, and hence converges to a function f_u in $\mathcal{C}(\mathbb{T})$. Moreover, show that $f_u = f$ a.e.

   (b) Show that if $f, g \in A(\mathbb{T})$, then their pointwise product $fg \in A(\mathbb{T})$ too.

   Note: it is quite simple to show that $A(\mathbb{T})$ is closed under scalar multiplication and pointwise sum $f+g$ as well. Hence $A(\mathbb{T})$ can be realised as a subalgebra of $\mathcal{C}(\mathbb{T})$, called the *Fourier algebra*. We note that $A(\mathbb{T})$ is point separating and $\bar{f} \in A(\mathbb{T})$ for any f in $A(\mathbb{T})$. Thus the *Stone-Weierstrass Theorem* tells us that $A(\mathbb{T})$ is uniformly dense in $\mathcal{C}(\mathbb{T})$. (Why isn't it all of $\mathcal{C}(\mathbb{T})$?)

   We say f is *piecewise differentiable*, if it is differentiable except at finitely may points. Then f' is defined a.e. on $[-\pi,\pi]$. Let
$$\mathcal{D}(\mathbb{T}) = \left\{ f \in \mathcal{C}(\mathbb{T}) : \begin{array}{c} f \text{ is piecewise differentiable and} \\ f' \text{ is bounded on its domain} \end{array} \right\}$$

   (c) If $f \in \mathcal{D}(\mathbb{T})$, show that f' is measurable on its domain, and integrable with
$$\int_{-\pi}^{\pi} f' = 0$$

   [Hint: f' can be written a.e. as a pointwise limit of a sequence of continuous functions $n[(1/n)*f - f]$; carefully use MVT to show that f is Lipschitz, and thus LDCT can be used to get to result.]

   Note: In PM451 you will see that $\int_a^b f' = f(b) - f(a)$ for any absolutely continuous function $f : [a,b] \to \mathbb{R}$. This result is *Lebesgue's Differentiation Theorem.* This theorem is used in the proof that a.e. x in $[a,b]$ is a Lebesgue point for f', which we did not cover in class.

(d) If $f \in \mathcal{D}(\mathbb{T})$, then it has Fourier coefficients

$$c_0(f') = 0 \quad \text{and} \quad c_n(f') = inc_n(f) \quad \text{for } n \in \mathbb{Z} \setminus \{0\}.$$

[Hint: (c) justifies "integration by parts".]

(e) Show that $\mathcal{D}(\mathbb{T}) \subset A(\mathbb{T})$.

[Hint: if $n \neq 0$, $|c_n(f)| = \frac{1}{|n|}|nc_n(f)|$; use (c) above and the Cauchy-Schwarz inequality to get an upper bound on their sum.]

We might well consider (e) to be a "Global Dini's Theorem", since, by (a), it tells us that if $f \in \mathcal{D}(\mathbb{T})$, then $\lim_{n\to\infty} \|s_n(f) - f\|_\infty = 0$. Examples of elements of $\mathcal{D}(\mathbb{T})$ are such fuctions as in 1 (d), above, or a "saw tooth", $f(t) = |t|$ on $[-\pi, \pi]$, continued 2π-periodically to $\mathbb{R}$.

3. Let $\mathcal{X}$ be an inner-product space. A sequence of vectors $\{f_k\}_{k=1}^\infty$ in $\mathcal{X}$ is called *linearly independant* if for each n in $\mathbb{N}$, the finite subset $\{f_k\}_{k=1}^n$ is linearly independant. We denote $\text{span}\{f_k\}_{k=1}^n = \{\sum_{k=1}^n \alpha_k f_k : \alpha_k \in \mathbb{C}, k = 1, \ldots, n\}$ and call ths the *linear span* of $\{f_k\}_{k=1}^n$.

(a) *Gram-Schmidt procedure.* If $\{f_k\}_{k=1}^\infty$ is a linearly independant set in $\mathcal{X}$, define a sequence $\{e_k\}_{k=1}^\infty$ recursively by

$$e_1 = \frac{1}{\|f_1\|} f_1 \text{ and } e_k' = f_k - \sum_{j=1}^{k-1} \langle f_k, e_j \rangle e_j, \; e_k = \frac{1}{\|e_k'\|} e_k' \text{ for } k > 1.$$

Show that $\{e_k\}_{k=1}^\infty$ is an orthonormal sequence which satisfies $\text{span}\{e_k\}_{k=1}^n = \text{span}\{f_k\}_{k=1}^n$ for each n.

(b) Show that the inner-product space $\mathcal{X}$ is separable if and only if it admits an orthonormal basis sequence.

4. The *Haar system.* Define a sequence of intervals

$$I_0 = [0, 1], \; I_{n,k} = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right) \; (n \in \mathbb{N}, k = 1, \ldots, 2^n - 1), \; I_{n,2^n} = \left[\frac{2^n - 1}{2^n}, 1\right]$$

and then a sequence of elements of $L_2[0, 1]$ by

$$\psi_0 = \chi_{I_0}, \; \psi_{n,j} = 2^{(n-1)/2}\left(\chi_{I_{n,2j-1}} - \chi_{I_{n,2j}}\right) \text{ for } n \in \mathbb{N}, j = 1, \ldots, 2^{n-1}.$$

(a) Show that $\{\psi_0, \psi_{n,j} : n \in \mathbb{N}, j = 1, \ldots, 2^{n-1}\}$ is an ortho-normal system in $L_2[0, 1]$.

(b) Show that if $\varphi \in E_n = \text{span}\{\chi_{I_{n,k}}\}_{k=1}^{2^n}$ — we might call φ a *dyadic step function* of order n — then

$$H_n(\varphi) = \varphi, \text{ where } H_n(f) = \langle f, \psi_0 \rangle \psi_0 + \sum_{m=1}^{n} \sum_{j=1}^{2^{m-1}} \langle f, \psi_{m,j} \rangle \psi_{m,j} \text{ for } f \text{ in } L_2[0, 1].$$

(c) Show that $\{\psi_0, \psi_{n,j} : n \in \mathbb{N}, j = 1, \ldots, 2^{n-1}\}$ is an ortho-normal basis for $L_2[0, 1]$. Deduce that $\lim_{n\to\infty} \|H_n(f) - f\|_2 = 0$ for f in $L_2[0, 1]$.

[Hint: You can show directly that elements of $L_2[0, 1]$ can be approximated by dyadic step functions; or that elements of $C[0, 1]$ are uniformly approximated by such, then use A4, Q1.]