Pure Math 450, Assignment 6

Sample Solutions

1. **(a)** Let $k \neq 0$. By direct calculation, using that $e^{-ikt} = \cos kt - i \sin kt$ and that the integral of an odd function in a symmetric interval is zero (why?) we get

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt = c_{-k}(f).$$

Moreover, we find by evenness that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\chi_{[-\pi,0]}(t) + \chi_{[0,\pi]}(t)\right] f(t) \cos kt \, dt = \frac{1}{\pi} \int_{0}^{\pi} f(t) \cos kt \, dt$$

Thus, we obtain

$$\begin{aligned} s_n(f,t) &= \sum_{k=-n}^{n} c_k(f) e^{ikt} = c_0(f) + \sum_{k=1}^{n} \left(c_{-k}(f) e^{-ikt} + c_k(f) e^{ikt}\right) \\ &= c_0(f) + \sum_{k=1}^{n} c_k(f)(e^{-ikt} + e^{ikt}) = c_0(f) + 2 \sum_{k=1}^{\infty} c_k(f) \cos kt. \end{aligned}$$

(b) We have $c_0(f) = \frac{1}{\pi} \int_0^{\pi/2} 1 ds = \frac{1}{2}$ and for $k \neq 0$ we have

$$c_k(f) = \frac{1}{\pi} \int_0^{\pi/2} \cos ks \, ds = \frac{\sin(k\pi/2)}{\pi k} = \begin{cases} 0 & \text{if } k \text{ is even} \\ \frac{(-1)^j}{2j+1} & \text{if } k = 2j+1. \end{cases}$$

(c) We have from (b) that

$$s_{2n}(f,t) = c_0(f) + \sum_{k=1}^{2n} c_k(f) \cos kt = \sum_{j=1}^{n-1} \frac{2(-1)^j}{\pi(2j+1)} \cos(2j+1)t.$$

Using any of Dini's Theorem, Hardy's Tauberian Theroem, or Fejer's Theorem combined with alternating series test, we see that

$$1 = f(0) = \lim_{n \to \infty} s_{2n}(f,0) = \frac{1}{2} + \sum_{j=0}^{\infty} \frac{2(-1)^j}{\pi(2j+1)}$$

hence $\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} = \pi/4$.

[We may verify that $\{1, \sqrt{2}\cos(k\cdot)\}_{k=1}^{\infty}$ is an orthonormal basis for $L_2^e(\mathbb{T}) = \{f \in L_2(\mathbb{T}) : f(-t) = f(t)$ for a.e. $t\}$. Indeed we first use the fact that $2\cos kt \cos lt = \frac{1}{2}(e^{i(k+l)t} + e^{i(k-l)t} + e^{i(l-k)t} + e^{-i(k+l)t})$ to show that this sequence is indeed orthonormal. Now if $f \in L_2^e(\mathbb{T})$ then by Riesz-Fischer $\lim_{n\to\infty} \|f - s_n(f)\|_2 = 0$, so by (a), span$\{1, \sqrt{2}\cos(k\cdot)\}_{k=1}^{\infty}$ is dense in $L_2^e(\mathbb{T})$.] [Alternatively, we may recall from (a) that $c_{-k}(f) = c_k(f)$, so $\sum_{k=-n}^{n} |c_k(f)|^2 = |c_0|^2 + 2\sum_{k=1}^{n} |c_k(f)|^2$] Either way, we can immediately use Bessel's identity to see that

$$\frac{1}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^2 = \|f\|_2^2 = |c_0(f)|^2 + \sum_{k=1}^{\infty} 2|c_k(f)|^2 = \frac{1}{4} + \frac{2}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^2}$$

and hence $\sum_{j=0}^{\infty} \frac{1}{(2j+1)^2} = \pi^2/8$. Finally, if $S = \sum_{k=1}^{\infty} \frac{1}{k^2}$, then by standard manipulations with converging sums we have

$$S - \frac{\pi^2}{8} = \sum_{j=1}^{\infty} \frac{1}{(2j)^2} = \frac{S}{4} \qquad \Rightarrow \qquad S = \frac{\pi^2}{6}.$$

(d) We have $c_0(g) = \frac{1}{\pi}\int_0^{\pi} \cosh \alpha s \, ds = \frac{\sinh \alpha\pi}{\alpha\pi}$. If $k \geq 1$ we do 2 steps of integration by parts to get

$$\begin{aligned} I_k = \int_0^{\pi} \cosh \alpha s \cos ks \, ds &= \frac{\sinh \alpha\pi \cdot (-1)^k}{\alpha} + \frac{k}{\alpha}\int_0^{\pi} \sinh \alpha s \sin ks \, ds \\ &= \frac{\sinh \alpha\pi \cdot (-1)^k}{\alpha} - \frac{k^2}{\alpha^2} I_k. \end{aligned}$$

Solve for I_k and divide by π to get

$$c_k(g) = \frac{\alpha \sinh \alpha\pi}{\pi}\frac{(-1)^k}{\alpha^2 + k^2}$$

(e) From (d) we obtain

$$s_n(g,t) = \frac{\sinh \alpha\pi}{\alpha\pi} + \frac{2\alpha \sinh \alpha\pi}{\pi}\sum_{k=1}^{n}\frac{(-1)^k}{\alpha^2 + k^2}\cos kt.$$

Using Dini's Theorem, Hardy's Tauberian Therorem, the integral test for series and Fejer's Theorem, or q. 2 below, we see that

$$1 = g(0) = \frac{\sinh \alpha\pi}{\alpha\pi} + \frac{2\alpha \sinh \alpha\pi}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{\alpha^2 + k^2}$$

and hence

$$\frac{\alpha\pi - \sinh \alpha\pi}{2\alpha^2 \sinh \alpha\pi} = \sum_{k=1}^{\infty}\frac{(-1)^k}{\alpha^2 + k^2}.$$

Similarly (using the Lipschitz version of Dini's theorem instead of Dini's Theorem, or any of the other techniques) we have

$$\cosh \alpha\pi = g(\pi) = \frac{\sinh \alpha\pi}{\alpha\pi} + \frac{2\alpha \sinh \alpha\pi}{\pi}\sum_{k=1}^{\infty}\frac{1}{\alpha^2 + k^2}$$

and hence

$$\frac{\alpha\pi\cosh\alpha\pi - \sinh\alpha\pi}{2\alpha^2\sinh\alpha\pi} = \sum_{k=1}^{\infty}\frac{1}{\alpha^2+k^2}.$$

[As an exercise, take $\alpha \to 0$ on the left hand side (expand numerator and denominator into Taylor Series to make this easier) to obtain $\frac{\pi^2}{6}$. By the Weirestrauss M-Test, the series of functions on the right converges uniformly in any disc about 0, and thus defines a continuous function in such a disc.] As in (c) above we use Bessel's identity to obtain

$$\frac{\sinh 2\alpha\pi}{2\alpha\pi} + 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cosh^2\alpha s\, ds = \frac{\sinh^2\alpha\pi}{\alpha^2\pi^2} + \frac{2\alpha^2\sinh^2\alpha\pi}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(\alpha^2+k^2)^2}$$

from which we can extract a(n unattractive) formula for $\sum_{k=1}^{\infty}\frac{1}{(\alpha^2+k^2)^2}$.

2. **(a)** Let $f_n = \sum_{k=-n}^{n} c_k(f)e^k$ Then $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy; check that for $m < n$ we have

$$\|f_n - f_m\|_\infty = \sum_{k=-n}^{-(m+1)}|c_k(f)| + \sum_{k=m+1}^{n}|c_k(f)|.$$

Thus this sequence has a uniform limit f_u. Check that $c_n(f_u) = c_n(f)$ for each n from which it follows $f_u = f$ a.e. (by Corollary to Abstract Summability Kernel Theorem). [Notice, this is really the Weirestrauss M-Test.]

(b) By (a), above, we may work in $\mathcal{C}(\mathbb{T})$, which is uniformly complete. Write $f = \sum_{k=-\infty}^{\infty} c_k(f)e^k$, $g = \sum_{j=-\infty}^{\infty} c_j(g)e^j$, where the sums are regarded as converging (absolutely) uniformly. We not that multiplication by a fixed element is both linear and continuous on $\mathcal{C}(\mathbb{T})$ so we have

$$\begin{aligned} fg &= f\cdot\sum_{j=-\infty}^{\infty} c_j(g)e^j = \sum_{j=-\infty}^{\infty} c_j(g)fe^j \\ &= \sum_{j=-\infty}^{\infty} c_j(g)\left(\sum_{k=-\infty}^{\infty} c_k(f)e^k\right)e^j = \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} c_j(g)c_k(f)e^{k+j} \end{aligned}$$

What we would like to do, right now, is re-arrange our sums over fixed functions e^l. Set $l = k+j$ and the candidate coefficient for e^l is $\sum_{k=-\infty}^{\infty} c_{l-k}(g)c_k(f)$. This makes sense as

$$\begin{aligned} \sum_{l=-\infty}^{\infty}\left|\sum_{k=-\infty}^{\infty} c_{l-k}(g)c_k(f)\right| &\le \sum_{l=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|c_{l-k}(g)||c_k(f)| \qquad (\dagger) \\ &= \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty}|c_{l-k}(g)|\right)|c_k(f)| = \sum_{k=-\infty}^{\infty}\left(\sum_{l=-\infty}^{\infty}|c_l(g)|\right)|c_k(f)| \\ &= \sum_{l=-\infty}^{\infty}|c_l(g)|\cdot\sum_{k=-\infty}^{\infty}|c_k(f)| < \infty \end{aligned}$$

Note that (†) is a valid comparison, even if the right hand side diverges; also since $\sum_{k=-\infty}^{\infty}|c_k| = \sup_{n\in\mathbb{N}}\sum_{k=-n}^{n}|c_k|$, the interchange of sums is really and interchange of suprema, and is always valid. Thus, appealing to the re-arrangements lemma below [$Z\times\mathbb{Z}$ replacing $\mathbb{N}$], we obtain

$$fg = \sum_{m=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} c_{m-k}(g)c_k(f)\right)e^m.$$

Since for a fixed m, $|c_m(h)| \leq \|h\|_1 \leq \|h\|_\infty$ for h in $\mathcal{C}(\mathbb{T})$ we have that $h \mapsto c_m(h)$ is continuous so $c_m(fg) = \sum_{k=-\infty}^{\infty} c_{k-m}(g)c_k(f)$, and $\big(c_m(fg)\big)_{m\in\mathbb{Z}}$ is summable.

[**Lemma.** (Absolutley summing series allow arbitrary re-arrangements.) *If $\mathcal{X}$ is a Banach space, $\{x_k\}_{k=1}^{\infty}$ is a sequence in $\mathcal{X}$ such that $\sum_{k=1}^{\infty}\|x_k\| < \infty$, then $x = \lim_{n\to\infty}\sum_{k=1}^{n} x_k$ exists (as usual we write $x = \sum_{k=1}^{\infty} x_k$) and for any bijection $\sigma : \mathbb{N} \to \mathbb{N}$ we have*

$$\lim_{n\to\infty}\sum_{k=1}^{n} x_{\sigma(k)} \text{ exists and equals } x.$$

Proof. The first statement in is proved in (a), above. To see the second, let $\varepsilon > 0$. Find n such that $\sum_{k=n+1}^{\infty}\|x_k\| < \varepsilon/2$, and let $m = \max\{\sigma(1), \ldots, \sigma(n)\}$. Then check that for $m' \geq m$ we have $\left\|x - \sum_{k=1}^{m'} x_{\sigma(k)}\right\| < \varepsilon$.]

(c) Write $f'(t) = \lim_{n\to\infty} n\big(f(t+1/n) - f(t)\big)$ for a.e. $t \in \mathbb{R}$. Note this makes f' the pointwise limit of continuous functions. Let $\{s_1 < \cdots < s_m\} \subset [-\pi, \pi]$ be the finite set of points of non-differentiability of f, and $M = \sup\{|f'(t)| : t \neq s_i \text{ for any } i = 1, \ldots, m\}$. Let us use this to show that f is Lipschitz. First, if $s_j < s < t < s_{j+1}$ (here $s_{m+1} = s_1 + 2\pi$), then by MVT

$$|f(s) - f(t)| = |f'(\xi)||s-t| \leq M|s-t|.$$

[In $\mathbb{C}$-valued case we have $|f(s) - f(t)| = [|\mathrm{Re}\, f(s) - \mathrm{Re}\, f(t)|^2 + |\mathrm{Im}\, f(s) - \mathrm{Im}\, f(t)|^2]^{1/2} = [|(\mathrm{Re}\, f)'(\xi_1)|^2|s-t|^2 + |(\mathrm{Im}\, f)'(t)|^2|s-t|^2]^{1/2} \leq \sqrt{2}M|s-t|$, but let's not quibble over a bounding constant.] If, $s_j < s < s_{j+1}$ (here $s_{m+1} = s_1 + 2\pi$) then by continuity of f

$$|f(s) - f(s_j)| = \lim_{t\to s_j^+}|f(s) - f(t)| \leq \lim_{t\to s_j^+} m|s-t| = M|s-s_j|.$$

Similarly $|f(s_j) - f(s_{j+1})| \leq M|s_j - s_{j+1}|$. Now if $s_{j-1} < s \leq s_j < s_k < t < s_{k+1}$ for some $1 \leq j < k < 2m$ (here we employ the convention that $s_0 = s_m - 2\pi$ and $s_k = s_{k-m} + 2\pi$ for $m+1 \leq k < 2m$), we have

$$\begin{aligned}|f(s) - f(t)| &= |f(s) - f(s_j)| + \sum_{i=j}^{k-1}|f(s_i) - f(s_{i+1})| + |f(s_k) - f(t)| \\ &\leq M|s-s_j| + \sum_{i=j}^{k-1} M|s_i - s_{i+1}| + M|s_k - t| = M|s-t|\end{aligned}$$

Hence we find that

$$n|f(t+1/n)+f(t)| \le nM|(t+1/n)-t| = M$$

So the constant function M is an integrable majorant for $(-\frac{1}{n})*f - f$. Thus by LDCT, and then translation invariance we get

$$\int_{-\pi}^{\pi} f' = \lim_{n\to\infty} n \int_{-\pi}^{\pi} \big((-1/n)*f - f\big) = 0.$$

(d) That $c_0(f') = 0$ follows immediately. Now let $g = fe^{-n}$. Then

$$g'(t) = f'(t)e^{-int} - inf(t)e^{-int}$$

so it follows that

$$0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g' = c_n(f') - inc_n(f).$$

(e) If $f \in \mathcal{D}(\mathbb{T})$, then $f' \in L_2(\mathbb{T})$ [i.e. there is $g \in L_2(\mathbb{T})$ so $g = f'$ a.e.]. Then we have

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} |c_n(f)| &= |c_0(f)| + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n(f)| = |c_0(f)| + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{|n|}|c_n(f')| \\
&= |c_0(f)| + \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{n^2}\right)^{1/2} \left(\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |c_n(f')|^2\right)^{1/2} \qquad \text{(by CS } \le\text{)} \\
&\le |c_0(f)| + 2\frac{\pi^2}{6}\,\|f'\|_2 \quad \text{(by Bessel's } \le\text{)} \\
&< \infty.
\end{aligned}$$

3. **(a)** By assumption (which?), $f_1 \neq 0$ so $e_1 \neq 0$. Let us suppose inductively that $e_1, \dots, e_{k-1}$ can be created as claimed. If $k \ge 2$ and $1 \le i < k$ we easily check

$$\langle e_k', e_i\rangle = \left\langle f_k - \sum_{j=1}^{k-1}\langle f_k, e_j\rangle e_j, e_i \right\rangle = 0$$

so e_k' is orthogonal to each $e_1, \dots, e_{k-1}$. Moreover

$$f_k = e_k' + \sum_{j=1}^{k-1}\langle f_k, e_j\rangle e_j \in \operatorname{span}\{e_j\}_{j=1}^k.$$

Thus, inductively, we find $\operatorname{span}\{e_1, \dots, e_{k-1}, e_k'\} = \operatorname{span}\{f_j\}_{j=1}^k$. Since $\{f_j\}_{j=1}^\infty$ is linearly independant, dim $\operatorname{span}\{f_j\}_{j=1}^k = k$ and we find that $e_k' \neq 0$. Thus $\{e_j\}_{j=1}^k$ is orthonormal. Hence $\{e_j\}_{j=1}^k$ is orthonormal too.

(b) Suppose $\mathcal{X}$ has a dense sequence $\{d_k\}_{k=1}^\infty$ then $\bigcup_{n=1}^\infty \text{span}\{d_k\}_{k=1}^n$ is dense in $\mathcal{X}$ (why?). We recursively find a linearly independant set as follows:

- let $n_1 = \min\{n \in \mathbb{N} : d_n \neq 0\}$;
- if $m > 1$, let $n_m = \min\big\{n \in \mathbb{N} : d_n \notin \text{span}\{d_{n_1}, \ldots, d_{n_{m-1}}\}\big\}$.

Now let $f_k = d_{n_k}$. If $\dim \mathcal{X} < \infty$, this process terminates after finitely many steps and we obtain a basis (why?); if not, this process produces an infinite sequence (why?). Verify that $\bigcup_{n=1}^\infty \text{span}\{f_k\}_{k=1}^n = \bigcup_{n=1}^\infty \text{span}\{d_k\}_{k=1}^n$. Now apply (a) to $\{f_k\}_{k=1}^\infty$ to obtain an orthonormal sequence $\{e_k\}_{k=1}^\infty$ for which $\text{span}\{e_k\}_{k=1}^\infty = \text{span}\{f_k\}_{k=1}^\infty$ is dense in $\mathcal{X}$.

Conversely, if $\mathcal{X}$ contains an orthonormal sequence $\{e_k\}_{k=1}^\infty$ for which $\text{span}\{e_k\}_{k=1}^\infty$ is dense in $\mathcal{X}$, then the countable set $\text{span}_{\mathbb{Q}[i]}\{e_k\}_{k=1}^\infty = \{\sum_{k=1}^n q_k e_k : n \in \mathbb{N}$ and $q_1, \ldots, q_n \in \mathbb{Q}[i]\}$ ($\mathbb{Q}[i] = \{q + ir : q, r \in \mathbb{Q}\}$ is the field of Gaussian rationals; we would use $\mathbb{Q}$ if we assume $\mathcal{X}$ is a $\mathbb{R}$-inner product space) is dense in $\text{span}\{e_k\}_{k=1}^\infty$. Indeed, if $f = \sum_{k=1}^n \alpha_k e_k \in \text{span}\{e_k\}_{k=1}^\infty$, then find $q_1, \ldots, q_n$ in $\mathbb{Q}[i]$ for which $|\alpha_k - q_k| < \varepsilon/\sqrt{n}$. Then

$$\left\| f - \sum_{k=1}^n q_k e_k \right\|^2 = \sum_{k=1}^n |\alpha_k - q_k|^2 < \varepsilon^2.$$

Since $\text{span}\{e_k\}_{k=1}^\infty$ is dense in $\mathcal{X}$, it follows a standard argument that $\text{span}_{\mathbb{Q}[i]}\{e_k\}_{k=1}^\infty$ is dense in $\mathcal{X}$ too.

[Note: since we do not assume $\mathcal{X}$ is complete, we must avoid using infinite sums. There is an abstract analogue of Riesz-Fischer: *A separable inner product space $\mathcal{X}$ is complete $\Leftrightarrow$ for any orthonormal basis $\{e_k\}_{k=1}^\infty$, $\sum_{k=1}^\infty \alpha_k e_k \in \mathcal{X}$ whenever $(\alpha_k)_{k=1}^\infty \in \ell_2(\mathbb{N})$.*]

4. **(a)** That $\|\psi_0\|_2 = 1$ and each $\|\psi_{n,j}\|_2 = 1$ is a rudimentary computation. Note that $\psi_{n,i}\overline{\psi_{n,j}} = 0$ for $i, j = 1, \ldots, 2^n$ with $i \neq j$, so $\langle \psi_{n,i}, \psi_{n,j} \rangle = 0$. Now if $m < n$, $i = 1, \ldots, 2^m$, $j = 1, \ldots, 2^n$ then

$$\psi_{m,i}\overline{\psi_{n,j}} = \begin{cases} 2^{(m+n-2)/2}\psi_{n,j} & \Leftrightarrow j = 2^{n-m}i - 1 \\ -2^{(m+n-2)/2}\psi_{n,j} & \Leftrightarrow j = 2^{m-n}i \\ 0 & \text{otherwise.} \end{cases}$$

In each case it is straightforward to verify $\langle \psi_{m,i}, \psi_{n,j} \rangle = 0$. Similarly, if $0 < n$ then $\langle \psi_0, \psi_{n,j} \rangle = 0$ for any $j = 1, \ldots, 2^n$.

(b) It is trivial to verify that $\{\chi_{I_{n,k}}\}_{k=1}^{2^n}$ is linearly independant, and thus is a basis for E_n. Moreover $\{\psi_0\} \cup \bigcup_{m=1}^{n-1}\{\psi_{m,j}\}_{j=1}^{2^m}$ is orthonormal, linearly independant, of cardinality $1 + 2 + \cdots + 2^{n-1} = 2^n$ and contained in E_n, so it must be an orthonormal basis for E_n. Thus by the (proof of) the Linear Approximation Lemma we find for φ in E_n

$$0 = \text{dist}(\varphi, E_n) = \|\varphi - H_n(\varphi)\|.$$

(c) **Solution #1.** Since measurable simple functions are dense in $L_2[0,1]$ by A4, Q2(a), it suffices to show for any measurable $E \subset [0,1]$ that χ_E can be approximated by dyadic step functions. Given $\varepsilon > 0$, find an open set G such that $E \subset G$ and $\lambda(G) < \lambda(E) + \sqrt{\varepsilon/2}$. Write $G \cap (0,1) = \dot{\bigcup}_{j\in\mathbb{N}}(a_j, b_j)$ (A1, Q4). Let for each j and n,

$$a'_{j,n} = \min\{k/2^n : k = 1, \ldots, 2^n - 1, a_i < k/2^n < b_i\} \text{ and}$$
$$b'_{j,n} = \max\{k/2^n : k = 1, \ldots, 2^n - 1, a_i < k/2^n < b_i\}.$$

Then $\varphi_n = \sum_{j=1}^{\infty} \chi_{[a'_{j,n}, b'_{j,n})}$ is a dyadic step function, and $\lim_{n\to\infty} \varphi_n = \chi_G$ a.e. (verify). Since $|\chi_G - \varphi_n|^2 \le \chi_G$, it follows the LDCT that $\lim_{n\to\infty} \|\chi_G - \varphi_n\|_2^2 = 0$. Thus there is n for which $\|\chi_G - \varphi_n\|_2^2 < \varepsilon^2/4$. Then it is a standard calculation that

$$\|\varphi_n - \chi_E\|_2 \le \|\varphi_n - \chi_G\|_2 + \|\chi_G - \chi_E\|_2 < \varepsilon.$$

Solution #2. Let $f \in L_2[0,1]$. Fix, for the moment, $m \in \mathbb{N}$. By A4, Q2, find $h \in \mathcal{C}[0,1]$ so $\|f - h\|_2 < 1/m$. Since h is uniformly continuous, there is n_m so $|s - t| < 2/2^{n_m}$ implies $|h(s) - h(t)| < 1/m$. Let $\varphi_{n_m} = \sum_{k=1}^{2^{n_m}} h(k/2^{n_m})\chi_{I_{n_m,k}}$. Then $\|\varphi_{n_m} - h\|_\infty < 1/m$. We thus have

$$\operatorname{dist}(f, E_{n_m}) \le \|f - \varphi\|_2 \le \|f - h\|_2 + \|h - \varphi\|_\infty < \frac{2}{m}$$

(why?). It is clear we can arrange $n_1 < n_2 < \ldots$, and thus

$$0 \le \lim_{n\to\infty} \operatorname{dist}(f, E_n) \le \lim_{m\to\infty} \operatorname{dist}(f, E_{n_m}) \le \lim_{m\to\infty} \frac{2}{m} = 0$$

which shows that $\bigcup_{n=1}^{\infty} \operatorname{span}\left(\{\psi_0\} \cup \bigcup_{l=1}^{n-1}\{\psi_{l,j}\}_{j=1}^{2^l}\right) = \bigcup_{n=1}^{\infty} E_n$ is dense in $L_2[0,1]$. [Notice, moreover, that $\lim_{n\to\infty} \|H_n(f) - f\|_2 = 0$ as by Linear Approximation Lemma, $\|H_n(f) - f\|_2 = \operatorname{dist}(f, E_n)$.]