PMath 451/651, Fall Term 2012

Homework Assignment 1 – Solutions

Problem 1 collects, in an abstract setting, the Boolean properties of half-open rectangles (or half-open intervals) which took us to the concept of algebra of sets.

Problem 1. Let X be a non-empty set and suppose that $\mathcal{F}$ is a collection of subsets of X which has the following properties:
(i) $X \in \mathcal{F}$.
(ii) If $F, G \in \mathcal{F}$ then $F \cap G \in \mathcal{F}$.
(iii) For every $F \in \mathcal{F}$ one can find $C_1, \ldots, C_n \in \mathcal{F}$ (for some $n \in \mathbb{N}$) such that $X \setminus F = C_1 \cup \cdots \cup C_n$.
Consider the collection $\mathcal{A}$ of subsets of X which is defined as follows:

$$\mathcal{A} := \{A \subseteq X \mid \exists\, n \geq 1 \text{ and } F_1, \ldots, F_n \in \mathcal{F} \text{ such that } F_1 \cup \cdots \cup F_n = A\}.$$

Prove that $\mathcal{A}$ is an algebra of subsets of X.

Solution.

- First, note that $X \in \mathcal{F} \subseteq \mathcal{A}$.
- If $A \in \mathcal{A}$, then by definition of $\mathcal{A}$, there are $F_1, \ldots, F_n \in \mathcal{F}$ such that $\bigcup_{i=1}^n F_i = A$, whence by property (iii) of $\mathcal{F}$, for each $i \in \{1, \ldots, n\}$, there are $C_{i,1}, \ldots, C_{i,k_i} \in \mathcal{F}$ such that $X \setminus F = \bigcup_{j=1}^{k_i} C_{i,j}$. Then

$$X \setminus A = X \setminus \bigcup_{i=1}^{n} F_i = \bigcap_{i=1}^{n} (X \setminus F_i) = \bigcap_{i=1}^{n} \bigcup_{j=1}^{k_i} C_{i,j} = \bigcup_{j \in \prod_{m=1}^{n}\{1,\ldots,k_m\}} \bigcap_{i=1}^{n} C_{i,j_i}.$$

  By property (ii) of $\mathcal{F}$ (applied $n-1$ times), each $\bigcap_{i=1}^n C_{i,j_i} \in \mathcal{F}$, whence by definition of $\mathcal{A}$, $\bigcup_{j \in \prod_{m=1}^n \{1,\ldots,k_m\}} \bigcap_{i=1}^n C_{i,j_i} \in \mathcal{A}$. Thus, $X \setminus A \in \mathcal{A}$.
- Finally, if $A, B \in \mathcal{A}$, then since the union of two finite unions is still a finite union, $A \cup B \in \mathcal{A}$.

Thus, $\mathcal{A}$ is an algebra of subsets of X. ∎

Problem 2. Let (X, d) be a metric space. Let $\mathcal{D}$ be the collection of all open subsets of X, and let us put

$$\mathcal{A} := \left\{ A \subseteq X \;\middle|\; \begin{array}{l} \exists n \geq 1 \text{ and } D_1, \ldots, D_n, C_1, \ldots, C_n \subseteq X \text{ such that } D_1, \ldots, D_n \text{ are open,} \\ C_1, \ldots, C_n \text{ are closed, and } A = (D_1 \cap C_1) \cup \cdots \cup (D_n \cap C_n) \end{array} \right\}.$$

Prove that $\mathcal{A}$ is the algebra of subsets of X which is generated by $\mathcal{D}$.

Solution. Let $\mathcal{F} = \{D \cap C \mid D \text{ is open and } C \text{ is closed}\}$. Then since X is both open and closed, letting $D = C = X$, we see that $X \in \mathcal{F}$. Also, if $D_1 \cap C_1, D_2 \cap C_2 \in \mathcal{F}$, then $(D_1 \cap C_1) \cap (D_2 \cap C_2) = (D_1 \cap D_2) \cap (C_1 \cap C_2)$, and since open sets stay open under finite intersections and closed sets stay closed under arbitrary intersections, we have that $(D_1 \cap C_1) \cap (D_2 \cap C_2) \in \mathcal{F}$. Now, if $D \cap C \in \mathcal{F}$, then $X \setminus (D \cap C) = (X \setminus D) \cup (X \setminus C)$. But, $X \setminus D$ is closed and $X \setminus C$ is open, so we see that $X \setminus (D \cap C) = C' \cup D'$ where $C' = X \setminus D$ and $D' = X \setminus C$. We can then write this as $X \setminus (D \cap C) = (C' \cap D_1) \cup (D' \cap C_1)$ where $C_1 = D_1 = X$ (since X is both open and closed). Thus, $\mathcal{F}$ satisfies all the conditions of Question 1. Therefore, by definition of $\mathcal{A}$, we have that $\mathcal{A}$ is an algebra of subsets of X.

Let $\mathcal{U}$ be any algebra of subsets of X containing $\mathcal{D}$. Then as before, $\mathcal{U}$ also contains all closed sets. Being closed under intersection, it also contains all subsets of the form $D \cap C$ where D is open and C is closed. Finally, since an algebra is closed under finite unions, we have for any collection of open sets $D_1, ..., D_n$ and any collection of closed sets $C_1, ..., C_n$ that $(D_1 \cap C_1) \cup ... \cup (D_n \cap C_n) \in \mathcal{U}$. Then for every $A \in \mathcal{A}$, we have $A \in \mathcal{U}$, so $\mathcal{A} \subseteq \mathcal{U}$. This shows that $\mathcal{A}$ is the smallest algebra of subsets of X containing $\mathcal{D}$; therefore, it must be the algebra of subsets of X generated by $\mathcal{D}$, and we are done. $\square$

Problem 3. In this problem we consider the metric space (X, d) where $X = [0, 1]$ and d is the usual distance on $[0, 1]$ (that is, $d(s, t) = |s - t|$ for $0 \leq s, t \leq 1$).

(a) Let $\mathcal{D}$ be the collection of all open subsets of X, and let $\mathcal{A}$ be the algebra of subsets generated by $\mathcal{D}$. Prove that $[0, 1] \cap \mathbb{Q} \notin \mathcal{A}$.

(b) Let $\mathcal{B}$ be the Borel σ-algebra of (X, d). Prove that $[0, 1] \cap \mathbb{Q} \in \mathcal{B}$.

3/3

Solution: For this problem let $Q = \mathbb{Q} \cap [0,1]$ and enumerate it as $Q = \{q_1, \dots, q_n, \dots\}$.

(a) The version of Baire's theorem that we need is this: "the countable intersection of dense open sets is dense". A set $\mathcal{U} \subseteq X$ is said to be of type G_δ if there exist open sets $D_1, D_2, \dots \subseteq X$ such that $\mathcal{U} = \bigcap_{n=1}^{\infty} D_n$.

Suppose Q were of type G_δ. Then there would exist open sets $D_1, D_2, \dots, D_n, \dots \subseteq X$ such that $Q = \bigcap_{n=1}^{\infty} D_n$.

Then $cl(Q) = cl(\mathbb{Q} \cap [0,1]) = cl(\mathbb{Q}) \cap cl([0,1]) = \mathbb{R} \cap [0,1] = [0,1] = X$ which means that, since $D_k \supseteq \bigcap_{n=1}^{\infty} D_n$ for every $k \in \mathbb{N}$, $cl(D_k) \supseteq cl(Q) = X$
$\Rightarrow cl(D_k) = X$

So the D_k, $k \geq 1$, are dense open sets.

Consider $E_n := X \setminus \{q_n\}$, $n \geq 1$. These E_n are dense ($cl(E_n) = cl(X \setminus \{q_n\}) = X \setminus int\{q_n\} = X \setminus \emptyset = X$), open sets too.

By Baire, $cl(D_1 \cap E_1 \cap D_2 \cap E_2 \cap \dots) = X$

But $D_1 \cap E_1 \cap D_2 \cap E_2 \cap \dots = \bigcap_{k=1}^{\infty} D_k \cap \bigcap_{n=1}^{\infty} E_n = Q \cap X \setminus Q = \emptyset$, whose closure is itself; a contradict!.

So Q is not of type G_δ.

Suppose $Q \in \mathcal{A}$. By Problem 2, $\exists n \geq 1$ and $\underset{open}{D_1, \dots, D_n}$, $\underset{closed}{C_1, \dots, C_n} \subseteq X$ such that $Q = (D_1 \cap C_1) \cup \dots \cup (D_n \cap C_n)$. The goal is to show that the RHS is a set of type G_δ.

Every open set is of type G_δ (just intersect it with itself, or X, countably many times)

Every closed set is of type G_δ, too! $F \text{ closed} = \bigcap_{n=1}^{\infty} \underbrace{\left(\bigcup_{x \in F} B(x, \tfrac{1}{n}) \right)}_{open}$ (from PM351)

Intersection of G_δ sets is obviously G_δ.

Finite unions are G_δ as well: Suppose $\mathcal{U}_n, V_n$, $n \geq 1$ are open subsets of X.

Then $\left(\bigcap_{n=1}^{\infty} \mathcal{U}_n\right) \cup \left(\bigcap_{m=1}^{\infty} V_m\right) = \bigcap_{n=1}^{\infty} \left(\mathcal{U}_n \cup \bigcap_{m=1}^{\infty} V_m\right) = \underbrace{\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty}}_{countable} \underbrace{(\mathcal{U}_n \cup V_m)}_{open}$. Hence Q, which is not G_δ, equals $(D_1 \cap C_1) \cup \dots \cup (D_n \cap C_n)$, which is G_δ, a contradiction. So $Q \notin \mathcal{A}$. □

NB.
$(A \cap B \cap C \cap \dots) \cup Z$
$\|$
$(A \cup Z) \cap (B \cup Z) \cap \dots$

(b) The Borel σ-algebra contains all closed sets, and is closed under countable union, so $Q = \bigcup_{n=1}^{\infty} \{q_n\} \in \mathcal{B}$. ∎

Problem 4. Let X be an infinite uncountable set, and let $\mathcal{U} := \{ \{x\} \mid x \in X\}$ (that is, $\mathcal{U}$ is the collection of all the 1-element subsets of X). Let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{U}$. Prove that $\mathcal{A}$ can be described as follows:

$$\mathcal{A} = \{B \subseteq X \mid B \text{ is countable}\} \cup \{C \subseteq X \mid X \setminus C \text{ is countable}\}.$$

Solution. Let us denote

$$\mathcal{B} := \{B \subseteq X \mid B \text{ is countable }\} \cup \{C \subseteq X \mid X \setminus C \text{ is countable }\}.$$

We prove by double inclusion that $\mathcal{B} = \mathcal{A}$.

"$\subseteq$" Every countable subset B of X must belong to $\mathcal{A}$, because B can be writen as a countable union of 1-element sets, and because $\mathcal{A}$ is closed under countable unions. The complements of countable subsets of X must then also be in $\mathcal{A}$, because $\mathcal{A}$ is closed under taking complements. Thus we have that $\mathcal{B} \subseteq \mathcal{A}$, as required.

"$\supseteq$" Let us start by verifying that $\mathcal{B}$ is a σ-algebra. For convenience we write $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where

$$\mathcal{B}_1 := \{B \subseteq X \mid B \text{ is countable }\} \text{ and } \mathcal{B}_2 := \{C \subseteq X \mid X \setminus C \text{ is countable }\}.$$

We check the three axioms for a σ-algebra.

(AS1) $X \in \mathcal{B}_2 \subseteq \mathcal{B}$, because the complement of X is $\emptyset$, which is countable.

(AS2) If $A \in \mathcal{B}$ then $X \setminus A \in \mathcal{B}$. Indeed, either A belongs to $\mathcal{B}_1$ in which case $X \setminus A$ belongs to $\mathcal{B}_2$, or vice-versa, A belongs to $\mathcal{B}_2$ in which case $X \setminus A$ belongs to $\mathcal{B}_1$.

(σ − AS3) Let $(A_n)_{n=1}^{\infty}$ be a countable family of sets from $\mathcal{B}$. If every A_n is from $\mathcal{B}_1$, then $\cup_{n=1}^{\infty} A_n$ is in $\mathcal{B}_1$ as well (a countable union of countable sets is still countable). In the opposite case there exists an index $n_o \geq 1$ such that $A_{n_o} \in \mathcal{B}_2$, and we have

$$X \setminus \left(\cup_{n=1}^{\infty} A_n\right) \subseteq X \setminus A_{n_o} \Rightarrow X \setminus \left(\cup_{n=1}^{\infty} A_n\right) \text{ is countable}$$
$$\Rightarrow \cup_{n=1}^{\infty} A_n \in \mathcal{B}_2.$$

So in any case, we obtain that $\cup_{n=1}^{\infty} A_n \in \mathcal{B}$. This completes the verification of the fact that $\mathcal{B}$ is a σ-algebra.

But now, it is clear that $\mathcal{B}$ contains $\mathcal{U}$ (as every 1-element set is countable, and thus belongs to $\mathcal{B}_1$). So we have

$$\left\{ \begin{array}{c} \mathcal{B} \text{ is a } \sigma\text{-algebra} \\ \mathcal{B} \supseteq \mathcal{U} \end{array} \right\} \Rightarrow \mathcal{B} \supseteq \mathcal{A},$$

with the last implication holding because $\mathcal{A}$ is the smallest σ-algebra which contains $\mathcal{U}$.

Problem 5. (a) Let X be a non-empty set, and let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two collections of subsets of X, such that $\mathcal{U}_1 \subseteq \mathcal{U}_2$. Let $\mathcal{A}_1$ be the σ-algebra generated by $\mathcal{U}_1$, and let $\mathcal{A}_2$ be the σ-algebra generated by $\mathcal{U}_2$. Prove that $\mathcal{A}_1 \subseteq \mathcal{A}_2$.

(b) Let (X, d) be a metric space. Let $\mathcal{D}_o$ be the collection of all the open balls in X (that is, $\mathcal{D}_o = \{B(a; r) \mid a \in X, r > 0\}$), and let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{D}_o$. Prove that $\mathcal{A} \subseteq \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra of (X, d).

(c) Show by example that the inclusion from part (b) of this question can be strict.

Solution.

a) By definition of σ-algebra generated by $\mathcal{U}_1$, we have that $\mathcal{U}_1 \subseteq \mathcal{U}_2 \subseteq \mathcal{A}_2$. Hence $\mathcal{A}_2$ is a σ-algebra contain $\mathcal{U}_1$. Also, we have $\mathcal{A}_1$is the smallest σ-algebra that contains $\mathcal{U}_1$, we must have $\mathcal{A}_1 \subseteq \mathcal{A}_2$. $\square$

b) As we proved in PMath 351, every open ball is a open set. We let $\mathcal{D}_1$be the collection of all the open sets in X. That is, $\mathcal{D}_1 = \{D \mid D \subset X,\ D \textit{ is open}\}$. We thus have $\mathcal{D}_0 \subseteq \mathcal{D}_1$. Thus, from result in a), we have $\mathcal{A} \subseteq \mathcal{B}$. $\square$

c) Consider metric space $(\mathbb{R}, d)$, where

$$d(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

We have unit interval $[0, 1]$ is an open set as $\forall x \in [0, 1],\ B(x, 0.5) = \{x\} \subset [0, 1]$. We have that $[0, 1] \in \mathcal{D}_1$.

We also note that, for any open ball centered at $a \in \mathbb{R}$, we have $B(a, r) = \begin{cases} \mathbb{R} & r > 1 \\ \{a\} & 0 < r \leq 1 \end{cases}$. We have $\mathcal{D}_0 = \{\{a \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\}\}$. From result in Problem 4, we have

$$\mathcal{A} = \{B \subseteq X \mid B \text{ is countable}\} \cup \{C \subseteq X \mid X \setminus C \text{ is countable}\}.$$

is σ-algebra generated by $\mathcal{D}_0$. But then, we have $[0, 1]$ is uncountable, nor is $\mathbb{R} \backslash [0, 1]$. We then have $[0, 1] \notin \mathcal{A}$. Indeed, we have $\mathcal{A} \subsetneq \mathcal{B}$. $\square$

In Problem 6 we use the concept of *separable* metric space. Recall from your previous analysis courses that a metric space (X, d) is said to be separable when there exists a countable subset $S \subseteq X$ such that S is dense in X.

Problem 6. Let (X, d) be a separable metric space. Let $\mathcal{D}_o$ be the collection of all the open balls in X, and let $\mathcal{A}$ be the σ-algebra generated by $\mathcal{D}_o$. Prove that $\mathcal{A} = \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra of (X, d).

Solution.

Let $\mathcal{D}$ be the collection of all open subsets of X. We will see that $\mathcal{D} \subseteq \mathcal{A}$. Let S be a dense countable subset of X and let $D \in \mathcal{D}$. For each $x \in D \cap S$, let

$$B_x := \bigcup\{B(x; r) \subseteq D : r > 0\}.$$

As D is open, we have that $B_x \neq \emptyset$, $B_x \subseteq D$ and $B_x \in \mathcal{D}_o$. We claim that

$$D = \bigcup_{x \in D \cap S} B_x. \quad (*)$$

Let $y \in D$. Then as D is open, there exists $\epsilon > 0$ such that $B(y; \epsilon) \subseteq D$. Since S is dense, there exists $x_0 \in S$ such that $d(x_0, y) \leq \epsilon/3$. Note that $x_0 \in B(y; \epsilon)$ and so $x_0 \in D \cap S$. Also, by the Triangle Inequality, we know that $B(x_0, 2\epsilon/3) \subseteq B(y; \epsilon) \subseteq D$ and so $B_{x_0} \supseteq B(x_0, 2\epsilon/3) \ni y$. As this is true for all $y \in D$, it follows that

$$D \subseteq \bigcup_{x \in D \cap S} B_x \subseteq D$$

and $(*)$ follows. However, as S is countable, so is $D \cap S$ and so D is in the σ-algebra generated by $\mathcal{D}_o$, i.e. $D \in \mathcal{A}$. Since this is true for all $D \in \mathcal{D}$, we obtain that $\mathcal{D} \subseteq \mathcal{A}$. Therefore, since $\mathcal{B}$ is the σ-algebra generated by $\mathcal{D}$, it follows that $\mathcal{B} \subseteq \mathcal{A}$. However, by 5)b we know that $\mathcal{A} \subseteq \mathcal{B}$ and so $\mathcal{A} = \mathcal{B}$. $\square$

Problem 7 *(restriction of an algebra/σ-algebra).*

Let X be a non-empty set, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mathcal{B}$ be the σ-algebra generated by $\mathcal{A}$. Let Y be a non-empty set which belongs to $\mathcal{A}$, and let us denote:

$$\mathcal{M} := \{M \mid M \in \mathcal{A},\ M \subseteq Y\}, \quad \mathcal{N} := \{N \mid N \in \mathcal{B},\ N \subseteq Y\}.$$

(a) Verify that $\mathcal{M}$ is an algebra of subsets of Y.
(b) Verify that $\mathcal{N}$ is a σ-algebra of subsets of Y.
(c) Prove that the σ-algebra generated by $\mathcal{M}$ is equal to $\mathcal{N}$.

[Note: It is common that the collections of sets $\mathcal{M}$ and $\mathcal{N}$ defined as above are denoted by $\mathcal{A}|_Y$ and $\mathcal{B}|_Y$, respectively.]

Solution. (a) We verify that $\mathcal{M}$ satisfies the three axioms defining an algebra of sets.

(AS1) Have $Y \in \mathcal{A}$ and $Y \subseteq Y$, hence $Y \in \mathcal{M}$.

(AS2) Let M be a set in $\mathcal{M}$. Then $Y \setminus M \in \mathcal{A}$ (because $Y, M \in \mathcal{A}$ and $\mathcal{A}$ is closed under set-difference), and $Y \setminus M \subseteq Y$. It follows that $Y \setminus M \in \mathcal{M}$.

(AS3) Let M and M' be two sets in $\mathcal{M}$. Then $M \cup M' \in \mathcal{A}$ (because $M, M' \in \mathcal{A}$ and $\mathcal{A}$ is closed under finite unions), and $M \cup M' \subseteq Y$ (because $M, M' \subseteq Y$). It follows that $M \cup M' \in \mathcal{M}$.

(b) We verify that $\mathcal{N}$ satisfies the three axioms defining a σ-algebra.

(AS1) $Y \in \mathcal{B}$ and $Y \subseteq Y$, hence $Y \in \mathcal{N}$.

(AS2) Let N be a set in $\mathcal{N}$. Then $Y \setminus N \in \mathcal{B}$ (because $Y, N \in \mathcal{B}$ and $\mathcal{B}$ is closed under set-difference), and $Y \setminus N \subseteq Y$. It follows that $Y \setminus N \in \mathcal{N}$.

(σ-AS3) Let $(N_n)_{n=1}^{\infty}$ be a countable family of sets from $\mathcal{N}$. Then $\cup_{n=1}^{\infty} N_n \in \mathcal{B}$ (because every N_n belongs to $\mathcal{B}$, and $\mathcal{B}$ is closed under countable unions), and $\cup_{n=1}^{\infty} N_n \subseteq Y$ (because $N_n \subseteq Y$ for all $n \geq 1$). It follows that $\cup_{n=1}^{\infty} N_n \in \mathcal{N}$.

(c) Let $\mathcal{P}$ denote the σ-algebra of subsets of Y which is generated by $\mathcal{M}$. We have to prove that $\mathcal{P} = \mathcal{N}$.

We first observe that the inclusion $\mathcal{P} \subseteq \mathcal{N}$ is immediate. Indeed, from, the given definitions is clear that $\mathcal{M} \subseteq \mathcal{N}$. Since we saw in (b) that $\mathcal{N}$ is a σ-algebra, it follows that $\mathcal{N}$ must contain the σ-algebra generated by $\mathcal{M}$ (which is $\mathcal{P}$).

The remaining part of the solution will be devoted to the proof of the reverse inclusion, $\mathcal{P} \supseteq \mathcal{N}$. We start on this by considering the collection $\widetilde{\mathcal{B}}$ of subsets of X defined by

$$\widetilde{\mathcal{B}} := \{B \in \mathcal{B} \mid B \cap Y \in \mathcal{P}\}.$$

We will prove the following claims about $\widetilde{\mathcal{B}}$.

Claim 1. $\widetilde{\mathcal{B}}$ is a σ-algebra.

Verification of Claim 1. We verify that $\widetilde{\mathcal{B}}$ satisfies the three axioms defining a σ-algebra.

(AS1) $X \cap Y = Y \in \mathcal{P}$, hence $X \in \widetilde{\mathcal{B}}$.

(AS2) Let $\widetilde{B}$ be a set in $\widetilde{\mathcal{B}}$. We want to prove that $X \setminus \widetilde{B}$ belongs to $\widetilde{\mathcal{B}}$ as well. This amounts to proving that $(X \setminus \widetilde{B}\,) \cap Y \in \mathcal{P}$. And indeed, we have

$$(X \setminus \widetilde{B}\,) \cap Y = Y \setminus \widetilde{B} = Y \setminus (Y \cap \widetilde{B}),$$

and the latter set is in $\mathcal{P}$ (because $Y \cap \widetilde{B} \in \mathcal{P}$, and $\mathcal{P}$ is closed under taking complements in Y).

(σ-AS3) Let $(\widetilde{B}_n)_{n=1}^{\infty}$ be a countable family of sets from $\widetilde{\mathcal{B}}$. We want to prove that $\cup_{n=1}^{\infty} \widetilde{B}_n$ belongs to $\widetilde{\mathcal{B}}$ as well. This amounts to proving that $\big(\cup_{n=1}^{\infty} \widetilde{B}_n\big) \cap Y \in \mathcal{P}$. And indeed, we have

$$\big(\cup_{n=1}^{\infty} \widetilde{B}_n\big) \cap Y = \cup_{n=1}^{\infty} \big(\widetilde{B}_n \cap Y\big),$$

and the latter set is in $\mathcal{P}$ (because $\widetilde{B}_n \cap Y$ belongs to $\mathcal{P}$ for every $n \geq 1$, and $\mathcal{P}$ is closed under countable unions).

Claim 2. $\widetilde{\mathcal{B}} \supseteq \mathcal{A}$.

Verification of Claim 2. If $A \in \mathcal{A}$, then from the definition of $\mathcal{M}$ it is clear that $A \cap Y \in \mathcal{M}$. But $\mathcal{M} \subseteq \mathcal{P}$; hence $A \cap Y \in \mathcal{P}$, which means that $A \in \widetilde{\mathcal{B}}$.

Claim 3. $\widetilde{\mathcal{B}} = \mathcal{B}$.

Verification of Claim 3. $\widetilde{\mathcal{B}}$ is a σ-algebra which contains $\mathcal{A}$, so it must contain the σ-algebra generated by $\mathcal{A}$. This means that $\widetilde{\mathcal{B}} \supseteq \mathcal{B}$, and the opposite inclusion is clear from how $\widetilde{\mathcal{B}}$ is defined.

Now back to proving the inclusion $\mathcal{P} \supseteq \mathcal{N}$. Claim 3 gives us that $B \cap Y \in \mathcal{P}$, for every $B \in \mathcal{B}$. But then let N be an arbitrary set in $\mathcal{N}$. We have $N \in \mathcal{B}$ and $N \cap Y = N$, hence we infer that $N = N \cap Y \in \mathcal{P}$ (which completes the verification of the required inclusion).