PMath 451/651, Fall Term 2012

Homework Assignment 2 – Solutions

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $(A_n)_{n=1}^{\infty}$ be a family of sets from $\mathcal{M}$. Consider the set

$$T := \left\{ x \in X \;\middle|\; \begin{array}{l} \text{for every } n \geq 1 \text{ there exists } k \geq n \\ \text{such that } x \in A_k \end{array} \right\}.$$

This set is sometimes called the "tail-set" of the sequence $A_1, A_2, \ldots, A_n, \ldots$.

(a) Prove that $T = \cap_{n=1}^{\infty} \big(\cup_{k=n}^{\infty} A_k \big)$, and that, as a consequence, T belongs to the σ-algebra $\mathcal{M}$.

(b) Suppose that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that $\mu(T) = 0$.

Solution.

(a) Let $x \in T$. Then for every $n \geq 1$, there exists $k \geq n$ such that $x \in A_k$. Thus, for every $n \geq 1$, we have $x \in \cup_{k=n}^{\infty} A_k$. Since x is in each of these unions, we see that it must be in the intersection of them as well. Thus, $x \in \cap_{n=1}^{\infty} (\cup_{k=n}^{\infty} A_k)$, and $T \subseteq \cap_{n=1}^{\infty} (\cup_{k=n}^{\infty} A_k)$. Now, let $x \in \cap_{n=1}^{\infty} (\cup_{k=n}^{\infty} A_k)$. Then for every $n \geq 1$, $x \in \cup_{k=n}^{\infty} A_k$. That is, for each $n \geq 1$, there exists a $k \geq n$ with $x \in A_k$. Thus, $x \in T$, and we see that $T = \cap_{n=1}^{\infty} (\cup_{k=n}^{\infty} A_k)$. Consequently, since $\mathcal{M}$ is closed under countable unions and countable intersections, we have that $T \in \mathcal{M}$.

(b) Let $\varepsilon > 0$. Since $\sum_{n=1}^{\infty} \mu(A_n) < \infty$, we can choose $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} \mu(A_n) < \varepsilon$. Now, we have the following:

$$\begin{aligned} \mu(T) &= \mu\left(\cap_{n=1}^{\infty} (\cup_{k=n}^{\infty} A_k)\right) \\ &\leq \mu\left(\cup_{k=N}^{\infty} A_k\right) \text{ since } T \subseteq \cup_{k=N}^{\infty} A_k \\ &\leq \sum_{k=N}^{\infty} \mu(A_k) \text{ by } \sigma\text{-subadditivity of } \mu \\ &< \varepsilon. \end{aligned}$$

Hence, since $\varepsilon > 0$ is arbitrary, we conclude that $\mu(T) = 0$. □

Problem 2 is related to the discussion around the Borel σ-algebra from Lecture 4. In this problem let us agree to use the following ad-hoc term: a subset $E \subseteq [0,1]$ will be called "$\mathbb{Q}$-efficient" when it has the following properties:

(E1) For every $t \in \mathbb{R}$, there exists an element $a \in E$ such that $t - a \in \mathbb{Q}$.

(E2) If a and b are distinct elements of E, then $a - b \notin \mathbb{Q}$.

Problem 2. In this problem $\mathcal{M}$ is a σ-algebra of subsets of $\mathbb{R}$ and $\mu : \mathcal{M} \to [0, \infty]$ is a positive measure. We assume that the measure space $(\mathbb{R}, \mathcal{M}, \mu)$ has the folowing properties:

- $[0,1]$ belongs to $\mathcal{M}$, and $\mu([0,1]) = 1$.
- (Invariance under translations). For every $A \in \mathcal{M}$ and $t \in \mathbb{R}$, one has that $A + t \in \mathcal{M}$ and $\mu(A+t) = \mu(A)$. (Here, as usual, $A + t := \{a + t \mid a \in A\}$.)

Prove that if $E \subseteq [0,1]$ is $\mathbb{Q}$-efficient, then $E \notin \mathcal{M}$.

Solution. Assume by contradiction that $E \in \mathcal{M}$. Then it makes sense to consider the quantity $\alpha := \mu(E) \in [0, \infty]$. We have in fact $0 \leq \alpha \leq 1$, with the second inequality holding because $E \subseteq [0,1]$, which implies that $\mu(E) \leq \mu([0,1]) = 1$.

Let us also consider an enumeration of the rationals in $[-1,1]$:

$$\mathbb{Q} \cap [-1,1] = \{q_1, q_2, \ldots, q_n, \ldots\},$$

and let us put $E_n := E + q_n$, $n \in \mathbb{N}$. We note that $\mu(E_n) = \alpha$, $\forall\, n \geq 1$, due to the fact that μ is translation-invariant.

By using the property (E2) for the set E, we observe that the sets E_n are pairwise disjoint. Indeed, assume by contradiction that we had an element $t \in E_m \cap E_m$ for some $m \neq n$. Then $a := t - q_m$ and $b := t - q_n$ are elements of E which satisfy $a - b = q_n - q_m \in \mathbb{Q}$ – contradiction with (E2). The union of the E_n's is therefore a disjoint union, which must have

$$(*) \qquad \mu\big(\cup_{n=1}^{\infty} E_n\big) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \alpha = \begin{cases} 0, & \text{if } \alpha = 0 \\ \infty, & \text{if } \alpha \neq 0. \end{cases}$$

On the other hand, by using the property (E1), we argue that that $\cup_{n=1}^{\infty} E_n \supseteq [0,1]$. Indeed, for every $t \in [0,1]$, property (E1) gives us an element $a \in E$ such that $t - a \in \mathbb{Q}$. We have $t - a \in [-1,1]$ (because $0 \leq a, t \leq 1$); hence $t - a$ must be equal to q_n for some $n \in \mathbb{N}$, and it follows that $t = a + q_n \in E_n$.

By applying μ to the inclusion $\cup_{n=1}^{\infty} E_n \supseteq [0,1]$ we find that $\mu\big(\cup_{n=1}^{\infty} E_n\big) \geq 1$. In order to get an inequality going the other way, we make the immediate observation that every E_n is a subset of $[-1,2]$ (this is because $E \subseteq [0,1]$ and $-1 \leq q_n \leq 1$, which implies that $-1 \leq a + q_n \leq 2$ for all $a \in E$). Hence $\cup_{n=1}^{\infty} E_n \subseteq [-1,2]$, and by applying μ to this inclusion we find that

$$\mu\big(\cup_{n=1}^{\infty} E_n\big) \leq \mu([-1,2]) \leq \mu([-1,0]) + \mu([0,1]) + \mu([1,2]) = 3.$$

The conclusion of the preceding paragraph is that $1 \leq \mu\big(\cup_{n=1}^{\infty} E_n\big) \leq 3$. But this is in contradiction with the formula obtained in $(*)$, which says that $\mu\big(\cup_{n=1}^{\infty} E_n\big)$ is equal to either 0 or ∞!

Hence the assumption that $E \in \mathcal{M}$ leads to contradiction, and it follows that $E \notin \mathcal{M}$, as required.

Problem 3 fills in the proof for an equivalent description of pre-measures which was mentioned in class (Remark 5.2).

Problem 3. Let X be a nonempty set, let $\mathcal{A}$ be an algebra of subsets of X, and let $\mu_o : \mathcal{A} \to [0, \infty]$ be an additive set-function. Consider the following two properties which μ_o may have.

$$\text{(Pre-}\sigma\text{-Add)} \quad \begin{cases} \text{Whenever } (A_n)_{n=1}^\infty \text{ are sets from } \mathcal{A} \text{ such that } A_i \cap A_j = \emptyset \text{ for } i \neq j \\ \quad \text{and such that } \cup_{n=1}^\infty A_n \text{ is still in } \mathcal{A}, \\ \text{it follows that } \mu_o(\cup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \mu_o(A_n). \end{cases}$$

$$\text{(Pre-}\sigma\text{-SubAdd)} \quad \begin{cases} \text{Whenever } A \text{ and } (A_n)_{n=1}^\infty \text{ are sets from } \mathcal{A} \text{ such that } A \subseteq \cup_{n=1}^\infty A_n, \\ \text{it follows that } \mu_o(A) \leq \sum_{n=1}^\infty \mu_o(A_n). \end{cases}$$

Prove that (Pre-σ-Add) and (Pre-σ-SubAdd) are equivalent to each other; that is, $\mathcal{A}$ has property (Pre-σ-Add) if and only if it has property (Pre-σ-SubAdd).

Solution. *Proof that "(Pre-σ-Add) $\Rightarrow$ (Pre-σ-SubAdd)".*

Let A and $(A_n)_{n=1}^\infty$ be sets from $\mathcal{A}$ such that $A \subseteq \cup_{n=1}^\infty A_n$. We want to prove that $\mu_o(A) \leq \sum_{n=1}^\infty \mu_o(A_n)$.

For every $n \geq 1$, let us put $A'_n := A_n \cap A$. Then $(A'_n)_{n=1}^\infty$ are sets from $\mathcal{A}$, with

$$\cup_{n=1}^\infty A'_n = \cup_{n=1}^\infty (A_n \cap A) = \left(\cup_{n=1}^\infty A_n \right) \cap A = A.$$

Let us next define $A''_1 = A'_1$ and $A''_n := A'_n \setminus \left(\cup_{i=1}^{n-1} A'_i \right)$, for $n \geq 2$. Then $(A''_n)_{n=1}^\infty$ also are sets from $\mathcal{A}$. It is easily seen that $A''_m \cap A''_n = \emptyset$ for $m \neq n$, and that $\cup_{n=1}^\infty A''_n = \cup_{n=1}^\infty A'_n = A$. The hypothesis (Pre-$\sigma$-Add) can be applied to the sets A''_n, and gives us that

$$\mu_o(A) = \sum_{n=1}^\infty \mu_o(A''_n).$$

But it is clear that $A''_n \subseteq A'_n \subseteq A_n$, hence that $\mu_o(A''_n) \leq \mu_o(A_n)$, $\forall\, n \geq 1$. So we conclude that

$$\mu_o(A) = \sum_{n=1}^\infty \mu_o(A''_n) \leq \sum_{n=1}^\infty \mu_o(A_n),$$

as we wanted.

Proof that "(Pre-σ-SubAdd) $\Rightarrow$ (Pre-σ-Add)".

Let $(A_n)_{n=1}^\infty$ be sets from $\mathcal{A}$ such that $A_m \cap A_n = \emptyset$ for $m \neq n$ and such that the union $A := \cup_{n=1}^\infty A_n$ still belongs to $\mathcal{A}$. We have to prove that $\mu_o(A) = \sum_{n=1}^\infty \mu_o(A_n)$.

The inequality $\mu_o(A) \leq \sum_{n=1}^\infty \mu_o(A_n)$ is provided by the property (Pre-σ-SubAdd) which we have here as hypothesis. Thus we only have to prove the opposite inequality, that

$$\mu_o(A) \geq \sum_{n=1}^\infty \mu_o(A_n). \tag{1}$$

The series on the right-hand side of (1) can be written as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} \mu_o(A_n) &= \lim_{N\to\infty} \Big(\sum_{n=1}^{N} \mu_o(A_n) \Big) \\ &= \lim_{N\to\infty} \mu_o(A_1 \cup \cdots \cup A_N)\end{aligned}$$

(where at the second equality sign we used the fact that μ_o is finitely additive). But for every $N \geq 1$ we have

$$\mu_o(A_1 \cup \cdots \cup A_N) \leq \mu_o(A), \tag{2}$$

because $A_1 \cup \cdots \cup A_N \subseteq A$, and μ_o is an increasing set-function. By making $N \to \infty$ in (2) we thus find that

$$\sum_{n=1}^{\infty} \mu_o(A_n) = \lim_{N\to\infty} \mu_o(A_1 \cup \cdots \cup A_N) \leq \mu_o(A),$$

as desired.

Problem 4. In this problem (X, d) is a metric space, $\mathcal{B}_X$ is the Borel σ-algebra of (X, d), and $\mu : \mathcal{B}_X \to [0, \infty)$ is a finite positive measure.

(a) Prove that for every $B \in \mathcal{B}_X$ and every $\varepsilon > 0$ there exist subsets $D, F \subseteq X$ such that
(i) D is open and F is closed;
(ii) $F \subseteq B \subseteq D$; and
(iii) $\mu(D \setminus F) < \varepsilon$.

(b) Prove that for every $B \in \mathcal{B}_X$ one has

$$\mu(B) = \inf\{\mu(D) \mid D \text{ open, } D \supseteq B\} = \sup\{\mu(F) \mid F \text{ closed, } F \subseteq B\}.$$

(c) Let $\nu : \mathcal{B}_X \to [0, \infty)$ be a finite positive measure which has the property that

$$\mu(D) = \nu(D) \text{ for every open subset } D \subseteq X.$$

Does it follow that $\mu = \nu$ (i.e. that $\mu(B) = \nu(B)$ for all $B \in \mathcal{B}_X$)? Justify your answer.

Solution. (a) Let us denote

$$\mathcal{F} := \left\{ B \in \mathcal{B} \;\middle|\; \begin{array}{l} \text{for every } \varepsilon > 0, \text{ there exist } D, F \subseteq X \\ \text{such that (i), (ii) and (iii) hold} \end{array} \right\}.$$

We have to prove that $\mathcal{F} = \mathcal{B}$. In view of the definition of $\mathcal{B}$, it will suffice to prove that $\mathcal{F}$ is a σ-algebra of subsets of X, and that $\mathcal{F}$ contains all the open sets. We divide the verifications of these facts into several claims.

Claim 1. Let G be an open subset of X. Then $G \in \mathcal{F}$.

Verification of Claim 1. Fix $\varepsilon > 0$. We have to find an open set D and a closed set F such that $F \subseteq G \subseteq D$ and such that $\mu(D \setminus F) < \varepsilon$.

We know that every open set is of type F_σ, hence there exists an increasing chain $(F_n)_{n=1}^\infty$ of closed subsets of X such that $\cup_{n=1}^\infty F_n = G$. From the continuity of μ along increasing chains it follows that $\lim_{n\to\infty} \mu(F_n) = \mu(G)$, so there exists n_o such that $\mu(F_{n_o}) > \mu(G) - \varepsilon$. We can then take $D = G$ and $F = F_{n_o}$, and all the required conditions will be satisfied (where for (iii) we write that $\mu(D \setminus F) = \mu(D) - \mu(F) = \mu(G) - \mu(F_{n_o}) < \varepsilon$).

Claim 2. If $B \in \mathcal{F}$, then $X \setminus B \in \mathcal{F}$.

Verification of Claim 2. Fix $\varepsilon > 0$. We have to find an open set D and a closed set F such that $F \subseteq X \setminus B \subseteq D$ and such that $\mu(D \setminus F) < \varepsilon$. From the hypothesis that $B \in \mathcal{B}$ we know that we can find an open set D' and a closed set F' such that $F' \subseteq B \subseteq D'$ and such that $\mu(D' \setminus F') < \varepsilon$. Let us put $D := X \setminus F'$ and $F := X \setminus D'$. Then D is open, F is closed, and by taking complements in the inclusions $F' \subseteq B \subseteq D'$ we find that $D \subseteq X \setminus B \supseteq F$. Moreover, some easy Boolean algebra shows that $D \setminus F = D' \setminus F'$; so we have

$$\mu(D \setminus F) = \mu(D' \setminus F') < \varepsilon,$$

and thus D and F have all the required properties.

Claim 3. If $B_1, B_2, \ldots, B_n \in \mathcal{F}$ (for some $n \in \mathbb{N}$), then $\cup_{i=1}^{n} B_i \in \mathcal{F}$.

Verification of Claim 3. Let us denote $\cup_{i=1}^{n} B_i =: B$. Fix $\varepsilon > 0$. We have to find an open set D and a closed set F such that $F \subseteq B \subseteq D$ and such that $\mu(D \setminus F) < \varepsilon$. From the hypothesis that every B_i is in $\mathcal{B}$ we infer the existence of open sets $D_1, \ldots, D_n$ and of closed sets $F_1, \ldots, F_n$ such that

$$F_i \subseteq B_i \subseteq D_i \text{ and } \mu(D_i \setminus F_i) < \varepsilon/n, \quad \text{for } 1 \leq i \leq n.$$

Let us put $D := \cup_{i=1}^{n} D_i$ and $F := \cup_{i=1}^{n} F_i$. Then D is open, F is closed, and $F \subseteq B \subseteq D$. Moreover, some easy Boolean algebra shows that

$$D \setminus F \subseteq \cup_{i=1}^{n} (D_i \setminus F_i),$$

which implies that

$$\mu(D \setminus F) \leq \sum_{i=1}^{n} \mu(D_i \setminus F_i) < n \cdot (\varepsilon/n) = \varepsilon.$$

Hence D and F have all the required properties.

Claim 4. If $(B_n)_{n=1}^{\infty}$ is an increasing chain of sets from $\mathcal{F}$, then $\cup_{n=1}^{\infty} B_n \in \mathcal{F}$.

Verification of Claim 4. Let us denote $\cup_{n=1}^{\infty} B_n =: B$. Fix $\varepsilon > 0$. We have to find an open set D and a closed set F such that $F \subseteq B \subseteq D$ and such that $\mu(D \setminus F) < \varepsilon$.

Due to the fact that μ is continuous along increasing chains, we have $\lim_{n \to \infty} \mu(B_n) = \mu(B)$, and hence we can find $n_o \in \mathbb{N}$ such that $\mu(B_{n_o}) > \mu(B) - \varepsilon/4$.

For every $n \geq n_o$, due to the hypothesis that $B_n \in \mathcal{F}$, we can find an open set D_n and a closed set F_n such that $F_n \subseteq B_n \subseteq D_n$ and such that $\mu(D_n \setminus F_n) < \varepsilon/4^n$. Let us define $D := \cup_{n=n_o}^{\infty} D_n$ and $F := F_{n_o}$. Then D is open and F is closed. We have moreover that

$$D \supseteq \cup_{n=n_o}^{\infty} B_n = B, \text{ and } F \subseteq B_{n_o} \subseteq B.$$

The verification of the claim will be completed if we can also show that $\mu(D \setminus F) < \varepsilon$. Since $\mu(D \setminus F) = \mu(D) - \mu(F)$, it will be sufficient to verify that

$$(\square) \qquad \mu(D) - \mu(B) < \varepsilon/2 \text{ and } \mu(B) - \mu(F) < \varepsilon/2.$$

For the first of the two inequalities ($\square$) we argue as follows:

$$\begin{aligned} D \setminus B &= \left(\cup_{n=n_o}^{\infty} D_n\right) \setminus \left(\cup_{n=n_o}^{\infty} B_n\right) \\ &\subseteq \cup_{n=n_o}^{\infty} (D_n \setminus B_n) \\ &\subseteq \cup_{n=n_o}^{\infty} (D_n \setminus F_n), \end{aligned}$$

hence

$$\mu(D) - \mu(B) = \mu(D \setminus B) \leq \sum_{n=n_o}^{\infty} \mu(D_n \setminus F_n) < \sum_{n=n_o}^{\infty} \frac{\varepsilon}{4^n} < \frac{\varepsilon}{2}.$$

For the second of the two inequalities ($\square$) we argue as follows:

$$\begin{aligned} \mu(B) - \mu(F) &= \mu(B) - \mu(F_{n_o}) \\ &= \big(\mu(B) - \mu(B_{n_o})\big) + \big(\mu(B_{n_o}) - \mu(F_{n_o})\big) \\ &< \varepsilon/4 + \big(\mu(D_{n_o}) - \mu(F_{n_o})\big) \\ &< \varepsilon/4 + \varepsilon/4^{n_o} \\ &\leq \varepsilon/2. \text{ (End of verification of Claim 4).} \end{aligned}$$

Claim 5. $\mathcal{F}$ is a σ-algebra of subsets of X.

Verification of Claim 5. We have $X \in \mathcal{F}$ from Claim 1 and the fact that X is an open set. The fact that $\mathcal{F}$ is closed under taking complements was proved in Claim 2. Finally, the fact that $\mathcal{F}$ is closed under countable unions follows from Claims 3 and 4. Indeed, let $(B_n)_{n=1}^{\infty}$ be a sequence of sets from $\mathcal{F}$. For every $n \geq 1$ let us put $C_n := \cup_{i=1}^{n} B_i$. Then $C_n \in \mathcal{F}$, $\forall\, n \geq 1$, due to Claim 3. Clearly, $(C_n)_{n=1}^{\infty}$ is an increasing chain of sets, so Claim 4 gives that $\cup_{n=1}^{\infty} C_n \in \mathcal{F}$. But it is obvious that $\cup_{n=1}^{\infty} C_n = \cup_{n=1}^{\infty} B_n$, so we have obtained that $\cup_{n=1}^{\infty} B_n \in \mathcal{F}$.

The verification of Claim 5 completes the proof of part (a) of the problem.

(b) Fix a set $B \in \mathcal{B}$, and denote

$$\begin{cases} \alpha := \sup\{\mu(F) \mid F \text{ closed, } F \subseteq B\}, \\ \\ \beta := \inf\{\mu(D) \mid D \text{ open, } D \supseteq B\}. \end{cases}$$

Since $\mu(B) \geq \mu(F)$ for every closed set $F \subseteq B$, we have $\mu(B) \geq \alpha$. Likewise, from the fact that $\mu(B) \leq \mu(D)$ for every open set D such that $D \supseteq B$, we infer that $\mu(B) \leq \beta$. We have to prove that the inequalities $\alpha \leq \mu(B) \leq \beta$ are in fact equalities.

In order to prove that $\mu(B) \leq \alpha$ and that $\mu(B) \geq \beta$, it suffices to prove that

$$(\diamondsuit) \qquad \mu(B) \leq \alpha + \varepsilon \text{ and } \mu(B) \geq \beta - \varepsilon, \text{ for every } \varepsilon > 0.$$

So fix an $\varepsilon > 0$ for which we prove that $(\diamondsuit)$ holds. From part (a) we know that we can find an open set D and a closed set F such that $F \subseteq B \subseteq D$ and such that $\mu(D \setminus F) < \varepsilon$. By using this D and F we then infer that

$$\beta \leq \mu(D) = \mu(D \setminus F) + \mu(F) \leq \varepsilon + \mu(B),$$

and on the other hand that

$$\mu(B) \leq \mu(D) = \mu(F) + \mu(D \setminus F) \leq \alpha + \varepsilon.$$

This completes the verification of $(\diamondsuit)$.

(c) Yes, it does follow that $\mu = \nu$. Indeed, for every set $B \in \mathcal{B}$ we use the statement of part (b) for μ and for ν, and we find that

$$\mu(B) = \inf\{\mu(D) \mid D \text{ open, } D \supseteq B\} = \inf\{\nu(D) \mid D \text{ open, } D \supseteq B\} = \nu(B).$$

The property of μ described in Problem 4(b) is called *closed regularity*. One also has a property which is simply called *regularity*, and is defined as follows.

Definition. Let (X,d) be a metric space, let $\mathcal{B}_X$ be the Borel σ-algebra of (X,d), and let $\mu : \mathcal{B}_X \to [0,\infty)$ be a finite positive measure. We say that μ is *regular* when the following happens: for every $B \in \mathcal{B}_X$ one has that

$$\mu(B) = \inf\{\mu(D) \mid D \text{ open, } D \supseteq B\} = \sup\{\mu(K) \mid K \text{ compact, } K \subseteq B\}.$$

Problem 5 adresses a situation when regularity is guaranteed to hold for every finite positive measure on $\mathcal{B}_X$.

Definition. A metric space (X,d) is said to be *σ-compact* when there exist compact subsets $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n \subseteq \cdots$ of X such that $\cup_{n=1}^{\infty} K_n = X$.

Problem 5. Let (X,d) be a metric space which is σ-compact, and let $\mathcal{B}_X$ be the Borel σ-algebra of (X,d). Prove that every finite positive measure $\mu : \mathcal{B}_X \to [0,\infty)$ is regular.

Solution. Let $\mu : \mathcal{B}_X \to [0,\infty)$ be a finite positive measure. We have to prove that, for every $B \in \mathcal{B}_X$, one has

$$\mu(B) = \inf\{\mu(D) \mid D \text{ open, } D \supseteq B\} \tag{3}$$

and

$$\mu(B) = \sup\{\mu(K) \mid K \text{ compact, } K \subseteq B\}. \tag{4}$$

The fact that (3) holds was proved in Problem 4, here we only have to verify (4).

So fix a set $B \in \mathcal{B}_X$ and an $\varepsilon > 0$; we have to prove that there exists a compact set $K \subseteq B$ such that $\mu(K) > \mu(B) - \varepsilon$.

In Problem 4 it was shown that μ has the closed regularity property; this implies, in particular, that we can find a closed set $F \subseteq B$ such that $\mu(F) > \mu(B) - \varepsilon/2$. For the compact sets $(K_n)_{n=1}^{\infty}$ mentioned in the statement of this problem (with $K_n \subseteq K_{n+1}, \forall\, n \geq 1$ and with $\cup_{n=1}^{\infty} K_n = X$) we then have that

$$K_1 \cap F \subseteq K_2 \cap F \subseteq \cdots \subseteq K_n \cap F \subseteq \cdots$$

and that

$$\cup_{n=1}^{\infty}(K_n \cap F) = \left(\cup_{n=1}^{\infty} K_n\right) \cap F = X \cap F = F.$$

The continuity of μ along increasing sequences implies that $\lim_{n\to\infty} \mu(K_n \cap F) = \mu(F)$, hence we can find $n_o \in \mathbb{N}$ such that

$$\mu(K_{n_o} \cap F) > \mu(F) - \varepsilon/2.$$

Then $K := K_{n_o} \cap F$ is a compact subset of B, with $\mu(K) > \mu(F) - \varepsilon/2 > \mu(B) - \varepsilon$. This completes the verification of (4).

Problem 6. Let (X, d) be a metric space, let $\mathcal{B}_X$ be the Borel σ-algebra of (X, d), and let $\mu : \mathcal{B}_X \to [0, \infty)$ be a finite positive measure which is regular. Prove that there exists an open set $U \subseteq X$, uniquely determined, such that

(i) $\mu(U) = 0$, and

(ii) whenever $D \subseteq X$ is open and has $\mu(D) = 0$, it follows that $D \subseteq U$.

[A comment related to this problem: properties (i) + (ii) show together that U is the "largest open set of measure 0" for μ. Its complement $X \setminus U$ is called the *support* of μ.]

Solution. There exist for sure some open subsets $D \subseteq X$ such that $\mu(D) = 0$ (for instance $D = \emptyset$ has these properties). Let us consider the collection of sets

$$\mathcal{U} := \{D \subseteq X \mid D \text{ is open, and } \mu(D) = 0\},$$

and let us put

$$U := \bigcup_{D \in \mathcal{U}} D.$$

Then U is an open subset of X (because an arbitrary colection of open sets is still open), and obviously has the property (ii) required in the statement of the problem.

In order to prove that the set U found above also has the required property (i), we first make the following observation.

Claim. $\mu(K) = 0$ for every compact set K such that $K \subseteq U$.

Verification of Claim. Fix a compact set K such that $K \subseteq U$. Then $\mathcal{U}$ is an open cover for K, hence (by compactness) there exist finitely many sets $D_1, \ldots, D_n \in \mathcal{U}$ such that $K \subseteq \cup_{i=1}^{n} D_i$. But then

$$\mu(K) \leq \sum_{i=1}^{n} \mu(D_i) = 0,$$

and we get that $\mu(K) = 0$, as we wanted.

With the above claim in hand, we return to finalize the proof that U has property (i), i.e. that $\mu(U) = 0$. Since it is given that the measure μ is regular, we have

$$\mu(U) = \sup\{\mu(K) \mid K \text{ compact, } K \subseteq U\}.$$

But the above claim says that every $\mu(K)$ considered in the sup is equal to 0; it follows that $\mu(U)$ must be equal to 0 as well.

We are left to prove that U is uniquely determined by the properties (i) and (ii). In other words, we have to prove that if U' is an open subset of X which has properties (i) and (ii), then $U = U'$. For such U' we have

$$\left(\begin{array}{c} U' \text{ open and} \\ \mu(U') = 0 \text{ (by property (i) for } U') \end{array} \right) \Rightarrow \left(\begin{array}{c} U' \subseteq U \text{ (by} \\ \text{property (ii) for } U) \end{array} \right).$$

The similar argument where the roles of U and U' are switched gives the inclusion $U \subseteq U'$, and we conclude that $U = U'$, as required.

In Problems 7 and 8 we use the metric space (X, d) where X is the set of all infinite sequences of 0's and 1's, and the distance on X is defined by the formula:

$$d(x,y) := \sum_{k=1}^{\infty} \frac{|x(k) - y(k)|}{2^k},$$

for $x = (x(1), x(2), \ldots, x(k), \ldots)$ and $y = (y(1), y(2), \ldots, y(k), \ldots)$ in X. (In general, for a sequence s in X, we will use the notation "$s(k)$" for the kth component of s.) We will accept without proof that (X, d) is a compact metric space.

For every $m \geq 1$ and $p_1, \ldots, p_m \in \{0, 1\}$ we will denote by $D_{p_1,\ldots,p_m}$ the subset of X defined as follows: $D_{p_1,\ldots,p_m} := \{x \in X \mid x(1) = p_1, \ldots, x(m) = p_m\}$. We will accept without proof that every set $D_{p_1,\ldots,p_m}$ is at the same time open and closed in X. (We say in short that $D_{p_1,\ldots,p_m}$ is a "clopen" subset of X.)

Problem 7. Let (X, d) be as above, and let us denote by $\mathcal{D}$ the collection of all the subsets $D_{p_1,\ldots,p_m}$ of X:

$$\mathcal{D} = \{\, D_{p_1,\ldots,p_m} \mid m \geq 1,\ p_1, \ldots p_m \in \{0,1\}\,\}.$$

(a) Prove that every open subset of X can be written as a union of sets from $\mathcal{D}$.
(b) Prove that the σ-algebra generated by $\mathcal{D}$ is the Borel σ-algebra of (X, d).

Solution.

(a) Let U be an open set in X, so for each $x \in U$, there is an open ball $B(x; r_x) \subseteq U$ with $r_x > 0$. Since each $r_x > 0$, there is a $n_x \in \mathbb{N}$ such that $\frac{1}{2^{n_x}} < r_x$. For each $x \in U$, let $D_x := D_{x(1),\ldots,x(n_x)} \in \mathcal{D}$. Note that each $x \in D_x$ and if $y \in D_x$, then

$$d(x,y) = \sum_{k=1}^{\infty} \frac{|x(k) - y(k)|}{2^k} = \sum_{k=n_x+1}^{\infty} \frac{|x(k) - y(k)|}{2^k} \leq \sum_{k=n_x+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n_x}} < r_x,$$

so $y \in B(x; r_x)$. Hence, each $D_x \subseteq B(x; r_x)$.
Therefore, $U = \bigcup_{x\in U}\{x\} \subseteq \bigcup_{x\in U} D_x \subseteq \bigcup_{x\in U} B(x; r_x) \subseteq U$, so $U = \bigcup_{x\in U} D_x$.
Thus, every open set in X is a union of sets from $\mathcal{D}$. ∎

(b) Note that $\mathcal{D} = \bigcup_{m=1}^{\infty} \bigcup_{p\in\{0,1\}^m} \{D_p\}$ is a countable union of finite sets, so $\mathcal{D}$ is countable. Let $\mathcal{A}$ denote the σ-algebra generated by $\mathcal{D}$. Let $\mathcal{B}_X$ denote the Borel σ-algebra of (X, d). By Problem 7a, any open set of X is a union of sets from $\mathcal{D}$. Since $\mathcal{D}$ is countable, it follows that every open set of X is a countable union of sets from $\mathcal{D}$, and hence is in $\mathcal{A}$. Thus, by definition of $\mathcal{B}_X$ being the smallest σ-algebra containing all open sets of X, we have that $\mathcal{B}_X \subseteq \mathcal{A}$. On the other hand, since every set in $\mathcal{D}$ is open, we have that $\mathcal{D} \subseteq \mathcal{B}_X$. By definition of $\mathcal{A}$ being the smallest σ-algebra containing all sets in $\mathcal{D}$, we have that $\mathcal{A} \subseteq \mathcal{B}_X$. Therefore, $\mathcal{A} = \mathcal{B}_X$. ∎

Problem 8. Let (X, d) be the same metric space as in Problem 7, and let $\mathcal{B}_X$ be the Borel σ-algebra of (X, d). By using the Caratheodory extension theorem, prove that there exists a probability measure $\mu : \mathcal{B}_X \to [0, 1]$ such that $\mu(D_{p_1,\dots,p_m}) = 1/2^m$ for every $m \geq 1$ and every $p_1, \dots, p_m \in \{0, 1\}$.

Solution.

To use Caratheodory's theorem, we first require an algebra of sets and an appropriate set function that can be shown to be a pre-measure.

Let's call sets of the from $D_{p_1,\dots,p_m}$, cylinders. First we can show that any two cylinders either contain each other or are disjoint:

Consider $D^1 = D_{p_1,\dots,p_m}$ and $D^2 = D_{q_1,\dots,q_l}$. Without loss of generality, $m \leq l$.

Suppose we have the relation: $D^1 \not\subset D^2$ and $D^2 \not\subset D^1$.

Let $x \in D^1 \cap D^2$. Then $x = (p_1, \dots, p_m, \dots.)$ and $x = (q_1, \dots, q_l, \dots)$. It follows then that $p_1 = q_1, \dots, p_m = q_m$. So, we actually have that $D^2 = D_{p_1,\dots,p_m,q_{m+1},\dots,q_l}$. But then we have $D^2 \subset D^1$ which is a contradiction to our assumption. Thus, we must have D^1, D^2 are disjoint.

Now, consider the set: $F = \{A \mid A \text{ is a cylinder}\} \cup \{X\} \cup \{\phi\}$

As our algebra, let's consider the family: $\mathcal{A} = \{\cup_{i=1}^n A_i \mid A_i \in F\}$.

To show that this family is indeed an algebra, we note that F satisfies the following two properties:

(1) Closure under finite intersections

(2) The complement of each element in F is a union of elements of F

To show (1): Let D^1, D^2 be cylinders. As shown above, their intersection is either D^1, D^2 or ϕ. Thus, $D^1 \cap D^2 \in F$. If we look at intersections with X or ϕ, we also get something in F again.

To show (2): Let $A \in F$. If $A = X$ or $A = \emptyset$, then $X \setminus A \in F$.

Now, suppose $A = D_{p_1,\dots,p_k}$. We then have that

$$X - A = D_{q_1} \cup D_{p_1,q_2} \cup D_{p_1,p_2,q_3} \cup \dots \cup D_{p_1,\dots,p_{k-1},q_k}$$

where q_i is the opposite of p_1. That is, if $p_i = 1$ then $q_i = 0$ and vice-versa.

Then we see that the complement of any cylinder is a union of cylinders.

By problem 1 on HW1, we have that $\mathcal{A}$ is an algebra.

Now, on $\mathcal{A}$, let us define the following function, $\mu_0 : \mathcal{A} \to [0, \infty)$ by:

$\mu_0(\cup_{i=1}^n D^i) = \sum_{i=1}^n 2^{-|D_i|}$

where $|D_i|$ is the length of $(p_1, \dots, p_k)$ in the expression $D_{p_1,\dots,p_k}$.

Let's also define $\mu_0(\emptyset) = 0$ and $\mu_0(X) = 1$.

Before proving that μ_0 is a pre-measure, we must show that it is well-defined.

To do this, we first notice that any element of $\mathcal{A}$ can be represented as $\cup_{i=1}^{n} D^i$ where the D^i are pairwise disjoint or we have that our element of $\mathcal{A}$ is just X or $\emptyset$.

What remains to be shown is that if $\cup_{i=1}^{n} D^i = \cup_{j=1}^{m} E^j$ where the D^i and E^j are cylinders, then

$$\mu_0(\cup_{i=1}^{n} D^i) = \mu_0(\cup_{j=1}^{m} E^j).$$

Consider $D^i \subset \cup D^j$. Where $D^i = D_{p_1,...,p_k}$. We must have that $D^i \subset \cup E^j$.

Then one of the following two situations hold.

(1) There exists $E^j \subset \cup E^j$ such that $D^i \subset E^j$.

(2) $D^i = \bigcup_{l=1}^{p} E^l$ where the $E^l \subset \bigcup_{j=1}^{m} E^j$. Also, the E^l satisfy:

$$E^l = D_{p_1,...,p_k,p_{k+1},...,p_{k+m}}$$

and the lengths of the sequences $(p_1, ..., p_{k+m})$ are all the same. We have that $p = 2^m$.

In case (1), we find that $2^{-|D^i|} \leq 2^{-|E^l|}$. That is, we find that $\mu_0(D^i) \leq \mu_0(E^l)$.

In case (2), we find that $2^{-|E^l|} = 2^{-(k+m)}$. Thus, we have that

$$\mu_0(E^l) = \sum_{l=1}^{2^m} \mu_0(E^l) = \sum_{l=1}^{2^m} 2^{-(k+m)} = 2^{-k} = \mu_0(D^l)$$

Thus, we find that: $\mu_0(\cup_{i=1}^{n} D^i) = \sum_{i=1}^{n} 2^{-|D_i|}$.

Using this, we find that:

$$\mu_0(\cup_{i=1}^{n} D^i) \leq \mu_0(\cup_{j=1}^{m} E^j).$$

But then, we note that starting with D or E was arbitrary. Repeating the same procedure with E and D interchanged, we find that: $\mu_0(\cup_{j=1}^{m} E^j) \leq \mu_0(\cup_{i=1}^{n} D^i)$. This gives us $\mu_0(\cup_{j=1}^{m} E^j) = \mu_0(\cup_{i=1}^{n} D^i)$.

Therefore, our set function μ_0 is indeed well defined.

In order to show that our μ_0 is a pre-measure on the algebra $\mathcal{A}$, we need to show that μ_0 satisfies a certain pre-measure criterion.

From a lemma shown in lecture, it suffices to verify:

For every $A \in \mathcal{A}$ we have that:

$$(1)\ \mu_0(A) = \inf\{\mu_0(U) \mid U \in \mathcal{A},\ A \subset int(U)\}$$

and

$$(2)\ \mu_0(A) = \sup\{\mu_0(H) \mid H \in \mathcal{A},\ \overline{H} \subset A \text{ and } H \text{ is compact}\}$$

To verify (1):

Let $A \in \mathcal{A}$. For now, suppose $A \neq X$ or $\emptyset$. Then $A = \cup_{i=1}^{n} D^i$. Since A is a union of open sets, A is also open. Thus, we have that $\mu_0(A) \geq \inf\{\mu_0(U) \mid U \in \mathcal{A},\ A \subset int(U)\}$. Now, suppose we have that $A \subset U \in \mathcal{A}$ (we have that $A \subset int(U)$ since both A and U are open). By the above work, when showing μ_0 is well defined, we must have that $\mu_0(A) \leq \mu_0(U)$. Taking the infimum over all U, we find that $\mu_0(A) \leq \inf\{\mu_0(U) \mid U \in \mathcal{A},\ A \subset int(U)\}$ and so we get equality.

If $A = X$, let U be such that $X \subset U$. The only possible choice for U is X. Thus, $\mu_0(X) = \inf\{\mu_0(U) \mid U \in \mathcal{A},\ A \subset int(U)\}$.

To verify (2):

Let $A \in \mathcal{A}$. Since A is closed and X is compact, A is also compact. Thus, we can take $H = A$. This gives us that $\mu_0(A) \leq \mu_0(A) = \sup\{\mu_0(H) \mid H \in \mathcal{A},\ \overline{H} \subset A$ and H is compact$\}$. To get the other inequality, let $H \in \mathcal{A}$ such that $\overline{H} \subset A$. Since $H \in \mathcal{A}$, it is closed, so therefore we may write $\overline{H} = H$. Again, by the above work, since $H \in \mathcal{A}$ and $H \subset A$ we have that $\mu_0(H) \leq \mu_0(A)$. Thus, taking the supremum over all such H we find that: $\sup\{\mu_0(H) \mid H \in \mathcal{A},\ \overline{H} \subset A$ and H is compact$\} \leq \mu_0(A)$. Thus, we have equality here as well.

Thus, μ_0 is a pre-measure on our algebra $\mathcal{A}$. As a result, by Caratheodory's theorem, μ_0 extends to a measure on a σ-algebra containing $\mathcal{A}$. Since this is an extension, we retain that $\mu(X) = \mu_0(X) = 1$ and $\mu(D^i) = \mu_0(D^i) = 2^{-|D_i|}$. So, μ is indeed a probability measure that is an extension of μ_0.

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