PMath 451/651, Fall Term 2012

Homework Assignment 3 – Solutions

Problem 1 fills in the proof of Lemma 6.8 from Lecture 6. Recall that $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra of $(\mathbb{R}, d)$, where d is the usual distance on $\mathbb{R}$. Recall moreover that in Lecture 6 we denoted by $\mathcal{E}$ the algebra of half-open intervals of $\mathbb{R}$. This is the collection of all finite unions of sets from $\mathcal{J}$, where we put

$$\mathcal{J} = \{\emptyset\} \cup \{(a, b] \mid a < b \text{ in } \mathbb{R}\} \cup \{(-\infty, b] \mid b \in \mathbb{R}\} \cup \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\mathbb{R}\}.$$

Problem 1. (a) Prove that $\mathcal{E} \subseteq \mathcal{B}_{\mathbb{R}}$.

(b) Let us put $\mathcal{U} := \{(-\infty, b] \mid b \in \mathbb{R}\}$. (Note that $\mathcal{U} \subseteq \mathcal{J}$; hence, by part (a), $\mathcal{U}$ is a subset of $\mathcal{E}$). Prove that the σ-algebra generated by $\mathcal{U}$ is equal to $\mathcal{B}_{\mathbb{R}}$.

(c) Prove that, as a consequence of parts (a) and (b), the σ-algebra generated by $\mathcal{E}$ is equal to $\mathcal{B}_{\mathbb{R}}$.

Solution.

(a) Since any element from $\mathcal{E}$ is a finite union of elements from $\mathcal{J}$ and that $\mathcal{B}_{\mathbb{R}}$ is closed under countable unions, it suffices to show that $\mathcal{J} \subseteq \mathcal{B}_{\mathbb{R}}$.

Let $X \in \mathcal{J}$. If $X = \emptyset$ or $X = (a, \infty)$ for some $a \in \mathbb{R}$, then X is open and is trivialy in $\mathcal{B}$. If $X = (-\infty, b]$ for some $b \in \mathbb{R}$, then X is the complement of $(a, \infty) \subseteq \mathcal{B}_{\mathbb{R}}$ and is therefore in $\mathcal{B}_{\mathbb{R}}$ as well. We are now only left with the case $X = (a, b]$ for some $a < b$ in $\mathbb{R}$. But in that case $X = \big(\mathbb{R} \setminus (b, \infty)\big) \cap (a, \infty)$, which is in $\mathcal{B}_{\mathbb{R}}$ since we only did complements and intersections of open sets. Therefore $\mathcal{J} \subseteq \mathcal{B}_{\mathbb{R}}$, hence $\mathcal{E} \subseteq \mathcal{B}_{\mathbb{R}}$.

(b) Let $\mathcal{C}$ be the σ-algebra generated by $\mathcal{U}$. Since $\mathcal{U} \subseteq \mathcal{E} \subseteq \mathcal{B}_{\mathbb{R}}$, from the previous assignment, we have that $\mathcal{C} \subseteq \mathcal{B}_{\mathbb{R}}$. Again, from the privious assigment, it suffices to show that all the open sets are in $\mathcal{C}$ to show that $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{C}$ and have $\mathcal{B}_{\mathbb{R}} = \mathcal{C}$ as desired. But if a set is open, it can be written as the countable union of open intervals, so it suffices to show that all open intervals are in $\mathcal{C}$ to get that all open sets are in $\mathcal{C}$. But, for any $a < b$ in $\mathbb{R}$, we have that $(a, b) = \Big(\mathbb{R} \setminus (-\infty, a]\Big) \cap \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$, which is in $\mathcal{C}$. So $\mathcal{B}_{\mathbb{R}} \subseteq \mathcal{C}$, so $\mathcal{B}_{\mathbb{R}} = \mathcal{C}$.

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(c) Since $\mathcal{U} \subseteq \mathcal{E}$, we have that the σ-algebra generated by $\mathcal{U}$, which by part (b) is equal to $\mathcal{B}_{\mathbb{R}}$, is a subset of the σ-algebra generated by $\mathcal{E}$. And since, by part (a), $\mathcal{E} \subseteq \mathcal{B}_{\mathbb{R}}$, we have that the σ-algebra generated by $\mathcal{E}$ is a subset of $\mathcal{B}_{\mathbb{R}}$. Therefore, the σ-algebra generated by $\mathcal{E}$ is equal to $\mathcal{B}_{\mathbb{R}}$.

In Problem 2, (X, d) is a metric space and $\varphi : X \to X$ is a homeomorphism (which means that φ is bijective, and that both φ and its inverse $\varphi^{-1} : X \to X$ are continuous).

Problem 2. (a) Let $\mathcal{B}_X$ be the Borel σ-algebra of (X, d). Prove that if $B \in \mathcal{B}_X$, then $\varphi(B) \in \mathcal{B}_X$.

(b) Let $\mu : \mathcal{B}_X \to [0, \infty)$ be a finite positive measure, and suppose that

$$\mu(\varphi(D)) = \mu(D) \quad \text{for every open set } D \subseteq X.$$

Prove that $\mu(\varphi(B)) = \mu(B)$ for every $B \in \mathcal{B}_X$.

Solution. a> Fix $B \in \mathcal{B}_X$, since $\varphi^{-1} : X \to X$ continuous then by corollary 7.5 then φ^{-1} is $(\mathcal{B}_X, \mathcal{B}_X)$-measurable thus by definition, $\varphi(B) \in \mathcal{B}_X$

(b) By Assignment 2 Problem 4b, for every $B \in \mathcal{B}_X$,

$$\mu(B) = \inf\{\mu(D) \mid D \text{ open }, D \supseteq B\}.$$

Since $\varphi^{-1}(D)$ is open for every open set D and conversely, for every open set U, $\varphi(U)$ is open and $\varphi^{-1}(\varphi(U)) = U$, φ^{-1} maps the set of all open subsets of X bijectively to itself, so we have

$$\mu(B) = \inf\{\mu(\varphi^{-1}(D)) \mid \varphi^{-1}(D) \text{ open }, \varphi^{-1}(D) \supseteq B\}.$$

By assumption, $\mu(\varphi^{-1}(D)) = \mu(\varphi(\varphi^{-1}(D))) = \mu(D)$. Also, $\varphi^{-1}(D)$ is open iff D is open, since both φ and φ^{-1} are continuous. Moreover, $D = \varphi(\varphi^{-1}(D)) \supseteq \varphi(B) \Leftrightarrow \varphi^{-1}(D) \supseteq B$, since φ is a bijection. Thus,

$$\mu(B) = \inf\{\mu(D) \mid D \text{ open }, D \supseteq \varphi(B)\}.$$

But again by Assignment 2 Problem 4b, we have that

$$\mu(\varphi(B)) = \inf\{\mu(D) \mid D \text{ open }, D \supseteq \varphi(B)\}.$$

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Hence, $\mu(B) = \mu(\varphi(B))$ for every $B \in \mathcal{B}_X$. ∎

For every $A \subseteq \mathbb{R}$ and $t \in \mathbb{R}$ we denote $A + t := \{a + t \mid a \in A\} \subseteq \mathbb{R}$ (the translation of A by t). Note that, as a special case of Problem 2(a), one has $A + t \in \mathcal{B}_{\mathbb{R}}$ whenever $A \in \mathcal{B}_{\mathbb{R}}$ and $t \in \mathbb{R}$ (where $\mathcal{B}_{\mathbb{R}}$ is the Borel σ-algebra of $\mathbb{R}$, same as in Problem 1).

We will denote the Lebesgue measure on $\mathbb{R}$ by λ. In the terminology used in Lecture 6, this is the Lebesgue-Stieltjes measure $\lambda : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ which is uniquely determined by the requirement that its centered Stieltjes function is $G_\lambda(t) = t$.

We will also use the (trivial) fact that positive measures can be amplified by scalars in $[0, \infty)$. That is: if $(X, \mathcal{M})$ is a measurable space, if $\mu : \mathcal{M} \to [0, \infty]$ is a positive measure, and if c is a constant in $[0, \infty)$, then one can define a new positive measure $c\,\mu : \mathcal{M} \to [0, \infty]$ by putting $(c\,\mu)(A) := c \cdot \mu(A), \ \ \forall\, A \in \mathcal{M}$.

Problem 3. Let $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ be a Lebesgue-Stieltjes measures which is translation-invariant, in the sense that $\mu(B + t) = \mu(B), \ \ \forall\, B \in \mathcal{B}_{\mathbb{R}}$ and $t \in \mathbb{R}$. Denote $\mu(\,(0,1]\,) =: c \in [0, \infty)$. Prove that $\mu = c \cdot \lambda$, where λ is the Lebesgue measure.

Solution

Let G_μ be the centered Stieltjes function of μ, so

$$G_\mu(t) = \begin{cases} \mu((0,t]) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -\mu((t,0]) & \text{if } t < 0 \end{cases}.$$

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Now, for each $n \in \mathbb{N}$,

$$\bigsqcup_{k=0}^{n-1} \left(\frac{k}{n}, \frac{k+1}{n} \right] = (0,1].$$

Since μ is translation invariant, μ assigns the same measure to each of the sets in the above disjoint union. Since $\mu((0,1]) = c$, it follows that $G_\mu(\frac{1}{n}) = \mu((0, \frac{1}{n}]) = \frac{c}{n}$. Now, for any $m \in \mathbb{N}$,

$$\begin{aligned} G_\mu\left(\frac{m}{n}\right) &= \mu\left(\left(0, \frac{m}{n}\right]\right) = \mu\left(\bigsqcup_{k=0}^{m-1} \left(\frac{k}{n}, \frac{k+1}{n}\right]\right) = \sum_{k=0}^{m-1} \mu\left(\left(\frac{k}{n}, \frac{k+1}{n}\right]\right) = \sum_{k=0}^{m-1} \mu\left(\left(0, \frac{1}{n}\right]\right) \\ &= c \cdot \frac{m}{n}. \end{aligned}$$

$$\begin{aligned} G_\mu\left(-\frac{m}{n}\right) &= -\mu\left(\left(-\frac{m}{n}, 0\right]\right) = -\mu\left(\bigsqcup_{k=0}^{m-1} \left(-\frac{k+1}{n}, -\frac{k}{n}\right]\right) = -\sum_{k=0}^{m-1} \mu\left(\left(-\frac{k+1}{n}, -\frac{k}{n}\right]\right) \\ &= -\sum_{k=0}^{m-1} \mu\left(\left(0, \frac{1}{n}\right]\right) = -c \cdot \frac{m}{n}. \end{aligned}$$

Thus, G_μ agrees with cG_λ on $\mathbb{Q}$. Since cadlag functions are continuous from the right, $G_\mu - cG_\lambda$ is still continuous from the right and is identically zero on $\mathbb{Q}$. However, for any $x \in \mathbb{R}$, there is a sequence of rationals $\{x_n\}_{n \in \mathbb{N}}$ converging to x from above, and since $(G_\mu - cG_\lambda)(x_n) = 0$ for all $n \in \mathbb{N}$, by continuity from the right, it follows that $(G_\mu - cG_\lambda)(x) = 0$. Thus, $G_\mu - cG_\lambda \equiv 0$, so $G_\mu \equiv cG_\lambda$, so $\mu = c\lambda$. ∎

In the Bonus Problem 4 we consider the compact metric space (X, d) which was used in Problems 7 and 8 of homework assignment 2. Recall that the elements of X are infinite sequences of 0's and 1's, and that for a sequence $x \in X$ we use the notation "$x(k)$" for the k-th component of x. We will look at the probability space $(X, \mathcal{B}_X, \mu)$, where $\mathcal{B}_X$ is the Borel σ-algebra of (X, d) and $\mu : \mathcal{B}_X \to [0, 1]$ is the probability measure constructed in Problem 8 of homework assignment 2.

Bonus Problem 4. Let $(X, \mathcal{B}_X, \mu)$ be as above. Let us agree to say that a sequence $x \in X$ is a "suspicious-1" sequence when it has the following property:

for every $\ell \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that $x(k+1) = x(k+2) = \cdots = x(k+\ell) = 1$.

Let $S \subseteq X$ be the set of all suspicious-1 sequences.

(a) Prove that $S \in \mathcal{B}_X$.

(b) Determine what is $\mu(S)$. (In words: calculate the probability that a random sequence of 0's and 1's is a suspicious-1 sequence.)

Solution.

(a) Let T_n denote the set of all elements such that every string of consecutive 1 has length at most n. We show that T_n is closed. Assume the contrary holds. And suppose that $\{x_k\}_{k=1}^{\infty} \subset T_n$ converges to some $x \notin T_n$. So x contains a string of consecutive 1's of length greater than n. So suppose that $x(t) = x(t+1) = \cdots x(t+n) = 1$ for some t. We note from previous assignment that $x_k \to x$ implies $x_n(k) \to x(k)$ pointwise. Now pick x_M close enough to x such the their first $t+n$ terms coincide. This is impossible since x contains a string of consecutive 1's of length $n+1$ but x_M does not. Hence T_n is closed.

We further let $T = \cup_{n=1}^{\infty} T_n$. We note that $T \in \mathcal{B}_X$ since each T_n is closed. Moreover, T is precisely the set of elements such that one can find some $l \in \mathbb{N}$ and no string of consecutive 1's has length more than l. That is $T = S^c$. It follows that $S \in \mathcal{B}_X$ since $T \in \mathcal{B}_X$.

(b) (Reference for generating series used in the proof: http://aofa.cs.princeton.edu/lectures/09Strings.pdf) We will assume basic knowledge of combinatorics and use notations from assignment 2. First let us define, for all possible $k, n \in \mathbb{N}$, $D_{n,k}$ to be the set of all element such that the first $k-1$ terms form a string with no n consecutive 1's, then followed by 0, and then a string of n number of 1's. Then we note that $D_{k,n} \cap D_{j,n} = \emptyset$ if $k \neq j$. Moreover,

$$T_n^c = D_{\underbrace{1, \ldots, 1}_{n}} \cup \bigcup_{k=1}^{\infty} D_{k,n}$$

The RHS is included in the LHS is clear. For each element of T_n^c, there exits some l such that the there are n consecutive 1's starting at the l-th positive. Take k to be the minimal of all such l, then this element lies in $D_{k,n}$. Finally, observe that the union above is disjoint. We note that

$$D_{k,n} = \bigcup D_{\underbrace{*}_{k-1}, 0, \underbrace{1, \ldots, 1}_{n}}$$

[Solution to Problem 4, continued]

where $*$ is some string of length $k-1$ with no n consecutive 1's. Again, note that the union above is disjoint. Since there are finitely many choice for $*$, we have

$$\mu(D_{k,n}) = c_{k,n}\frac{1}{2^{k+n}}$$

where $c_{k,n}$ is the number of ways of choosing $*$, that is, the number of strings of length $k-1$ with no n consecutive 1's. So let $S_n(x)$ be the generating functions for $c_{k,n}$ (ie $c_{k,n} = \langle x^n, S_n(x)\rangle$). We note that a binary string with no n consecutive 1's is either a binary string of length less than n containing all 1 or such a string followed by 0 and followed by a binary string with no n consecutive 1's. So if we take the weight function to be the length of the string, then

$$S_n(x) = (1 + x + \cdots + x^{n-1})(1 + xS_n(x))$$

Solve to get

$$S_n(x) = \frac{1 - x^n}{1 - 2x - x^n}$$

Then we note

$$\begin{aligned}
\mu(T_n^c) &= \mu(D_{1,\ldots,1}) + \sum \mu\left(\bigcup_{k=1}^{\infty} D_{k,n}\right) \\
&= \frac{1}{2^n} + \frac{1}{2^{n+1}} S_n(1/2) \\
&= \frac{1}{2^n} + 1 - \frac{1}{2^n} \\
&= 1
\end{aligned}$$

Now note that

$$S = \bigcap_{n=1}^{\infty} T_n^c$$

and moreover $T_n^c \supset T_{n+1}^c$ for all n since string of $n+1$ consecutive 1's certainly contains a string of n consecutive 1's. It follows since μ is a finite measure, we have

$$\mu(S) = \lim_{n\to\infty} \mu(T_n^c) = 1$$

□