PMath 451/651, Fall Term 2012

Homework Assignment 4 – Solutions

Problem 1. Let $(X, \mathcal{M})$ be a measurable space, and let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $\mathrm{Bor}(X, \mathbb{R})$. Consider the following subsets of X:

$A := \{x \in X \mid \limsup_{n\to\infty} f_n(x) = \infty\}$.

$B := \{x \in X \mid \limsup_{n\to\infty} f_n(x) = -\infty\}$.

$C := \{x \in X \mid \liminf_{n\to\infty} f_n(x) = \infty\}$.

$D := \{x \in X \mid \liminf_{n\to\infty} f_n(x) = -\infty\}$.

$E := \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\}$.

Prove that $A, B, C, D, E \in \mathcal{M}$.

Solution.

Argument for the set A. We will show, equivalently, that $X \setminus A \in \mathcal{M}$. It is immediate that an element $x \in X$ belongs to $X \setminus A$ if and only if the sequence $(f_n(x))_{n=1}^{\infty}$ is bounded above. So then we can write:

$$\begin{aligned} X \setminus A &= \{x \in X \mid \exists\, k \in \mathbb{N} \text{ such that } f_n(x) \le k \text{ for all } n \ge 1\} \\ &= \cup_{k=1}^{\infty} \{x \in X \mid \ f_n(x) \le k \text{ for all } n \ge 1\} \\ &= \cup_{k=1}^{\infty} \ \cap_{n=1}^{\infty} \{x \in X \mid f_n(x) \le k\} \\ &= \cup_{k=1}^{\infty} \ \cap_{n=1}^{\infty} f_n^{-1}\Big((-\infty, k] \Big). \end{aligned}$$

For every $n, k \in \mathbb{N}$ we have that $f_n^{-1}\Big((-\infty, k] \Big) \in \mathcal{M}$, because f_n is a Borel function and $(-\infty, k]$ is a Borel set. Hence $X \setminus A$ can be obtained by starting with sets from $\mathcal{M}$ and by performing countable unions and intersections. This implies that $X \setminus A \in \mathcal{M}$, as required.

Argument for the set B. We write

$$\begin{aligned} B &= \{x \in X \mid \limsup_{n\to\infty} f_n(x) = -\infty\} \\ &= \{x \in X \mid \lim_{n\to\infty} f_n(x) = -\infty\} \\ &= \cap_{k=1}^{\infty} \{x \in X \mid \exists\, n_o \in \mathbb{N} \text{ such that } f_n(x) < -k \text{ for all } n \ge n_o\} \\ &= \cap_{k=1}^{\infty} \Big(\cup_{n_o=1}^{\infty} \{x \in X \mid f_n(x) < -k \ \ \forall\, n \ge n_o\} \Big) \\ &= \cap_{k=1}^{\infty} \Big(\cup_{n_o=1}^{\infty} \cap_{n=n_o}^{\infty} f_n^{-1}((-\infty, -k)) \Big). \end{aligned}$$

We know that every set of the form $f_n^{-1}((-\infty, -k))$ is in $\mathcal{M}$, because the f_n's are Borel functions. Since B can be obtained from sets of this form by performing countable unions and intersections, it follows that B belongs to $\mathcal{M}$ as well.

Argument for the set C. The set C can be written in the form

$$C = \{x \in X \mid \limsup_{n\to\infty}(-f_n)(x) = -\infty\};$$

so the fact that $C \in \mathcal{M}$ follows from an argument identical to the one shown for the set B, but where we use the sequence of functions $(-f_n)_{n=1}^{\infty}$.

Argument for the set D. The set D can be written in the form

$$D = \{x \in X \mid \limsup_{n\to\infty}(-f_n)(x) = \infty\};$$

so the fact that $D \in \mathcal{M}$ follows from an argument identical to the one shown for the set A, but where we use the sequence of functions $(-f_n)_{n=1}^{\infty}$.

Argument for the set E. We have

$$\begin{aligned} E &= \{x \in X \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite } \} \\ &= \{x \in X \mid \text{the sequence } (f_n(x))_{n=1}^{\infty} \text{ is Cauchy}\} \\ &= \cap_{k=1}^{\infty}\{x \in X \mid \ \exists\, n_o \text{ such that } |f_m(x) - f_n(x)| < 1/k \text{ for all } m, n \geq n_o\} \\ &= \cap_{k=1}^{\infty}\Big(\cup_{n_o=1}^{\infty}\{x \in X \mid \ -1/k < f_m(x) - f_n(x) < 1/k, \ \ \forall\, m, n \geq n_o\} \Big) \\ &= \cap_{k=1}^{\infty}\Big(\cup_{n_o=1}^{\infty} \cap_{m,n=n_o}^{\infty} g_{m,n}^{-1}(\,(-1/k, 1/k)\,) \Big), \end{aligned}$$

where for every $m, n \in \mathbb{N}$ we denoted $f_m - f_n =: g_{m,n}$.

The functions $g_{m,n}$ are Borel (because the f_n's are so), hence that every set of the form $g_{m,n}^{-1}(\,(-1/k, 1/k)\,)$ is in $\mathcal{M}$. Since E can be obtained from sets of this form by performing countable unions and intersections, we conclude that E belongs to $\mathcal{M}$ as well.

Problem 2. Let $(X, \mathcal{M})$ be a measurable space, and let $(P_n)_{n=1}^{\infty}$ be a sequence of subsets of X such that

(i) $P_n \in \mathcal{M}, \quad \forall n \geq 1$;

(ii) $P_n \cap P_m = \emptyset$ for $n \neq m$; and

(iii) $\cup_{n=1}^{\infty} P_n = X$.

On the other hand let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from $\mathrm{Bor}(X, \mathbb{R})$. We create a new function $f : X \to \mathbb{R}$ by the following rule:

$$\begin{cases} \text{For every } x \in X \text{ we pick the unique } n \in \mathbb{N} \text{ such that } x \in P_n, \\ \text{and we define } f(x) := f_n(x). \end{cases}$$

Prove that $f \in \mathrm{Bor}(X, \mathbb{R})$.

[Comment related to Problem 2: The procedure used to define f is called "patching" – we are patching together the functions f_n, by using the partition of X into the sets P_n.]

Solution. We fix a Borel subset $S \subseteq \mathbb{R}$ for which we will prove that $f^{-1}(S) \in \mathcal{M}$. We observe that

$$\begin{aligned} f^{-1}(S) &= f^{-1}(S) \cap X \\ &= f^{-1}(S) \cap \Big(\cup_{n=1}^{\infty} P_n \Big) \\ &= \cup_{n=1}^{\infty} \Big(f^{-1}(S) \cap P_n \Big) \\ &= \cup_{n=1}^{\infty} \Big(f_n^{-1}(S) \cap P_n \Big); \end{aligned}$$

at the last equality sign we used that f coincides with f_n on P_n, which implies that

$$f^{-1}(S) \cap P_n = \{x \in P_n \mid f(x) \in S\} = \{x \in P_n \mid f_n(x) \in S\} = f_n^{-1}(S) \cap P_n.$$

For every $n \geq 1$ we have that $f_n^{-1}(S) \in \mathcal{M}$, because $f_n \in \mathrm{Bor}(X, \mathbb{R})$. Since it is given that $P_n \in \mathcal{M}$, we infer that $f_n^{-1}(S) \cap P_n$ is in $\mathcal{M}$ as well. Finally, since $\mathcal{M}$ is closed under countable unions, it follows that

$$f^{-1}(S) = \cup_{n=1}^{\infty} \Big(f_n^{-1}(S) \cap P_n \Big) \in \mathcal{M},$$

as we wanted.

Definition. Let $(X, \mathcal{M})$ be a measurable space.

1^o A function $f : X \to \mathbb{R}$ is said to be *simple* when it only takes finitely many values (the image $f(X)$ is a finite subset of $\mathbb{R}$). We will use the notation

$$\text{Bor}_s(X, \mathbb{R}) := \{f \in \text{Bor}(X, \mathbb{R}) \mid f \text{ is simple}\}.$$

2^o Let A be a subset of X. We will use the notation I_A for the indicator function of A. That is, $I_A : X \to \mathbb{R}$ is the function defined by

$$I_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in X \setminus A. \end{cases}$$

Problem 3. Let $(X, \mathcal{M})$ be a measurable space.

(a) Prove that $\text{Bor}_s(X, \mathbb{R})$ is a unital subalgebra of the algebra of functions $\text{Bor}(X, \mathbb{R})$.

(b) Let A be a subset of X. Prove that $I_A \in \text{Bor}_s(X, \mathbb{R})$ if and only if $A \in \mathcal{M}$.

(c) Let f be a function in $\text{Bor}_s(X, \mathbb{R})$, where f is not identically equal to zero. Prove that one can write f as a linear combination

$$f = \alpha_1 I_{A_1} + \cdots + \alpha_n I_{A_n},$$

where $n \geq 1$ and where all the conditions stated below are satisfied:
- $\alpha_1, \ldots, \alpha_n \in \mathbb{R} \setminus \{0\}$ are such that $\alpha_i \neq \alpha_j$ for $i \neq j$;
- $A_1, \ldots, A_n$ are non-empty sets from $\mathcal{M}$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

(d) Prove the equality $\text{Bor}_s(X, \mathbb{R}) = \text{span}\{I_A \mid A \in \mathcal{M}\}$ (where "span" is the notation for the linear span of a set of functions).

Solution. (a) We see that $\mathbf{1} = I_X$ is a constant function, hence continuous and hence in $\text{Bor}_s(X, \mathbb{R})$. Suppose that $f, g \in \text{Bor}_s(X, \mathbb{R})$. Then $f, g \in \text{Bor}(X, \mathbb{R})$ and by a proposition from class, $fg, f + g \in \text{Bor}(X, \mathbb{R})$. Also, by assumption, $|f(X)| = n < \infty$ and $|g(X)| = m < \infty$. It is clear that $|fg(X)| \leq nm < \infty$ and $|(f + g)(X)| \leq nm < \infty$, so fg and $f + g$ are simple. Hence, $fg, f + g \in \text{Bor}_s(X, \mathbb{R})$. Moreover, for $\alpha \in \mathbb{R}$, we see that $\alpha f \in \text{Bor}(X, \mathbb{R})$ from class, and clearly αf is simple as well, so $\alpha f \in \text{Bor}_s(X, \mathbb{R})$. Hence, $\text{Bor}_s(X, \mathbb{R})$ is a unital subalgebra of $\text{Bor}(X, \mathbb{R})$. $\square$

(b) ($\Rightarrow$) Since $I_A \in \text{Bor}_s(X, \mathbb{R})$, we have in particular that
$A = I_A^{-1}(\{1\}) = I_A^{-1}([1, \infty)) \in \mathcal{M}$ by a proposition from class. Hence, $A \in \mathcal{M}$.
($\Leftarrow$) Suppose that $A \in \mathcal{M}$. Then let $a \in \mathbb{R}$. If $a > 1$ then $I_A^{-1}([a, \infty)) = \emptyset \in \mathcal{M}$. If $0 < a \leq 1$, then $I_A^{-1}([a, \infty)) = A \in \mathcal{M}$. If $a \leq 0$, then $I_A^{-1}([a, \infty)) = X \in \mathcal{M}$, so $I_A \in \text{Bor}(X, \mathbb{R})$. Therefore, since I_A is clearly simple, we have $I_A \in \text{Bor}_s(X, \mathbb{R})$. $\square$

[Solution to Problem 3, continued]

(c). Since $f \in Bor_s(\mathcal{X}, \mathbb{R})$, f is simple and $f(\phi) = 0$.

Also $f \neq 0$, $\exists n \in \mathbb{N}$, $n \geq 1$ s.t. $f(\mathcal{X})/\{0\} = \{\alpha_1, \cdots, \alpha_n\}$ with $\alpha_i \neq \alpha_j$, $\forall i \neq j$, $\alpha_i \in \mathbb{R}$.

let $A_i = f^{-1}(\alpha_i)$, $1 \leq i \leq n$, then

As $f \in Bor(\mathcal{X}, \mathbb{R})$, $\{\alpha_i\} \in B_{\mathbb{R}}$, $A_i = f^{-1}(\{\alpha_i\}) \in \mathcal{M}$, $\forall i$.

Also, for $i \neq j$,
$$A_i \cap A_j = f^{-1}(\{\alpha_i\}) \cap f^{-1}(\{\alpha_j\}) = f^{-1}(\{\alpha_i\} \cap \{\alpha_j\}) = f^{-1}(\phi) = \phi$$

For every i, $1 \leq i \leq n$, $\exists x_i \in \mathcal{X}$ s.t. $f(x_i) = \alpha_i$ as $f(\mathcal{X})/\{0\} = \{\alpha_1, \cdots, \alpha_n\}$.

So, $x_i \in A_i \Rightarrow A_i$ is non-empty.

For $g := \alpha_1 I_{A_1} + \cdots + \alpha_n I_{A_n}$,

As $A_i \in \mathcal{M}$, $1 \leq i \leq n$, by (b), $I_{A_i} \in Bor_s(\mathcal{X}, \mathbb{R})$.

By (a), $g = \alpha_1 I_{A_1} + \cdots + \alpha_n I_{A_n} \in Bor_s(\mathcal{X}, \mathbb{R})$, since it is closed under algebra operations.

For any $x \in \mathcal{X}$: if $x \in A_i$, $1 \leq i \leq n$, $f(x) = \alpha_i$; $g(x) = 0 + \cdots + 0 + \alpha_i \cdot 1 + 0 + \cdots + 0 = \alpha_i = f(x)$,

if $x \in \mathcal{X}/\bigcup_{i=1}^n A_i$, $f(x) = 0$; $g(x) = 0 + \cdots + 0 = 0 = f(x)$.

$\Rightarrow f(x) = g(x)$, $\forall x \in \mathcal{X}$.

By Uniqueness in $Bor_s(\mathcal{X}, \mathbb{R})$, $f = g$ or $f = \alpha_1 I_{A_1} + \cdots + \alpha_n I_{A_n}$. ∎

1+3/3

(d). If $f = 0$, then $f = I_\phi$, $\phi \in \mathcal{M}$. $\Rightarrow f \in \text{span}\{I_A \mid A \in \mathcal{M}\}$

If f is not equal zero everywhere, by (c), we have $f = \sum_{i=1}^n \alpha_i I_{A_i} \in \text{span}\{I_A \mid A \in \mathcal{M}\}$

$\Rightarrow Bor_s(\mathcal{X}, \mathbb{R}) \subseteq \text{span}\{I_A / A \in \mathcal{M}\}$; Also, by (b), $I_A \in Bor_s(\mathcal{X}, \mathbb{R})$, as $A \in \mathcal{M}$, and by (a),

$Bor_s(\mathcal{X}, \mathbb{R})$ is a subalgebra $\Rightarrow \text{span}\{I_A \mid A \in \mathcal{M}\} \subseteq Bor_s(\mathcal{X}, \mathbb{R}) \Rightarrow Bor_s(\mathcal{X}, \mathbb{R}) = \text{span}\{I_A \mid A \in \mathcal{M}\}$. ∎

Let us consider again the space of simple Borel functions $\mathrm{Bor}_s(X, \mathbb{R})$ which appeared in Problem 3. Let us also recall that a function $f : X \to \mathbb{R}$ is said to be bounded when there exists $r > 0$ such that $|f(x)| < r$ for all $x \in X$. The next problem shows that every bounded Borel function can be written as a uniform limit of functions from $\mathrm{Bor}_s(X, \mathbb{R})$.

Problem 4. Let $(X, \mathcal{M})$ be a measurable space, let f be a bounded function in $\mathrm{Bor}(X, \mathbb{R})$, and let ε be a positive number. Prove that there exists a function $g \in \mathrm{Bor}_s(X, \mathbb{R})$ such that $|f(x) - g(x)| < \varepsilon$ for every $x \in X$.

Solution. We fix a constant $C > 0$ such that $|f(x)| < C$ for all $x \in X$. Let $n \in \mathbb{N}$ be such that $C/n < \varepsilon$, and let us partition the interval $(-C, C]$ into $2n$ consecutive half-open intervals $J_1, J_2, \ldots, J_{2n}$ of length C/n. That is:

$$J_1 = (-C, -C + \frac{C}{n}], J_2 = (-C + \frac{C}{n}, -C + \frac{2C}{n}], \ldots, J_{2n} = (-C + \frac{(2n-1)C}{n}, C].$$

For every $1 \leq k \leq 2n$ let us denote $A_k := f^{-1}(J_k) \subseteq X$. The sets $A_1, \ldots, A_{2n}$ are in $\mathcal{M}$, because f is a Borel function. We observe that

$$\cup_{k=1}^{2n} A_k = \cup_{k=1}^{2n} f^{-1}(J_k) = f^{-1}\Big(\cup_{k=1}^{2n} J_k\Big) = f^{-1}\Big((-C, C]\Big) = X.$$

Moreover, the sets A_k are pairwise disjoint:

$$A_j \cap A_k = f^{-1}(J_j) \cap f^{-1}(J_k) = f^{-1}(J_j \cap J_k) = f^{-1}(\emptyset) = \emptyset, \quad \text{for } j \neq k.$$

Consider the function $g \in \mathrm{Bor}_s(X, \mathbb{R})$ defined as

$$g := \sum_{k=1}^{2n} \Big(-C + \frac{kC}{n}\Big) I_{A_k}$$

(where I_{A_k} stands, as usual, for the indicator function of the set A_k). We claim that g has the property required in the problem. Indeed, let x be an arbitrary element of X. Since the sets $(A_k)_{k=1}^{2n}$ are mutually disjoint and cover X, there exists a unique k such that $x \in A_k$. Since A_k is defined as $f^{-1}(J_k)$, we thus have

$$f(x) \in J_k = (-C + \frac{(k-1)C}{n}, -C + \frac{kC}{n}].$$

On the other hand from the definition of g it follows that

$$g(x) = -C + \frac{kC}{n},$$

and it is then clear that

$$|f(x) - g(x)| \leq \frac{C}{n} < \varepsilon,$$

as required.