PMath 451/651, Fall Term 2012

Homework Assignment 5 – Solutions

Notation. Let $(X, \mathcal{M})$ be a measurable space. We denote

$$\mathrm{Bor}_b(X, \mathbb{R}) := \{f \in \mathrm{Bor}(X, \mathbb{R}) \mid f \text{ is a bounded function}\}.$$

It is immediate that $\mathrm{Bor}_b(X, \mathbb{R})$ is a unital subalgebra of the algebra of functions $\mathrm{Bor}(X, \mathbb{R})$. Problem 1 offers a trick which is sometimes useful when studying $\mathrm{Bor}_b(X, \mathbb{R})$, in the metric space framework.

Problem 1. Let (X, d) be a metric space, and consider the corresponding measurable space $(X, \mathcal{B}_X)$, where $\mathcal{B}_X$ is the Borel σ-algebra of (X, d). Suppose that $\mathcal{F}$ is a collection of bounded functions from X to $\mathbb{R}$, which has the following properties.

(i) Every bounded continuous function $f : X \to \mathbb{R}$ belongs to $\mathcal{F}$.

(ii) $\mathcal{F}$ is an algebra of functions. (That is: if $f, g \in \mathcal{F}$ then $fg \in \mathcal{F}$, and $\alpha f + \beta g \in \mathcal{F}$ for every $\alpha, \beta \in \mathbb{R}$.)

(iii) If $(f_n)_{n=1}^{\infty}$ is a sequence of functions in $\mathcal{F}$ which converges pointwise to a bounded function $f : X \to \mathbb{R}$, then it follows that $f \in \mathcal{F}$.

Prove that $\mathcal{F} \supseteq \mathrm{Bor}_b(X, \mathbb{R})$.

Solution.

<u>Claim 1:</u> $I_F \in \mathcal{F}$ for every closed $F \subseteq X$.

proof: Let $F \subseteq X$ be closed. For each $n \in \mathbb{N}$, let $D_n := \bigcup_{x \in F} B(x, \frac{1}{n})$ be open. Clearly, $F \subseteq D_n$ $\forall n \in \mathbb{N}$.
For each $n \in \mathbb{N}$, let $f_n : X \to \mathbb{R}$ be cts such that $0 \leq f(x) \leq 1$ $\forall x \in X$, $f(x) = 1$ if $x \in F$, $f(x) = 0$ if $x \in X \setminus D_n$. We will show that $f_n(x) \xrightarrow[n\to\infty]{} I_F(x)$ $\forall x \in X$.
If $x \in F$, then $I_F(x) - f_n(x) = 0$ $\forall n \in \mathbb{N}$; if $x \notin F$, then $d_F(x) := \inf\{d(x,a) \mid a \in F\} > 0$ since F is closed, and we can't pick $n_0 \in \mathbb{N}$ such that $\frac{1}{n} < d_F(x)$ $\forall n \geq n_0$, and so $x \notin D_n$ $\forall n \geq n_0$, hence $f_n(x) = 0$ $\forall n \geq n_0$.

(Cont'd on other side)

[Solution to Problem 1, continued]

Thus $f_n(x) \to I_F(X)$ $\forall x \in X$, and since each f_n is cts and bounded, $(f_n)_{n=1}^{\infty}$ is in $\mathcal{F}$. Then by property (iii) of $\mathcal{F}$, $I_F \in \mathcal{F}$.

<u>Claim 2:</u> $I_B \in \mathcal{F}$ $\forall B \in \mathcal{B}_X$.

proof: Consider $\mathcal{M} := \{A \subseteq X \mid I_A \in \mathcal{F}\}$. We show this to be a σ-algebra of subsets of X.

(AS 1) X is closed, so $I_X \in \mathcal{F} \Rightarrow X \in \mathcal{M}$.

(AS 2) Let $A \in \mathcal{M}$, $I_A \in \mathcal{F}$. We have that $I_{X \setminus A} = \mathbb{1} - I_A$, where $\mathbb{1}(x) = 1$ $\forall x \in X$, and since $\mathcal{F}$ is an algebra of functions and both $\mathbb{1}, I_A \in \mathcal{F}$, we get that $I_{X\setminus A} \in \mathcal{F}$. Hence $X \setminus A \in \mathcal{M}$.

(σ-AS3) First note that if $A, B \in \mathcal{M}$, then $I_{A\cup B} = I_A + I_B - I_{A\cap B} = I_A + I_B - I_A \cdot I_B \in \mathcal{F}$ because $\mathcal{F}$ is an algebra.

Let $(A_n)_{n=1}^{\infty}$ be in $\mathcal{M}$. By a trivial induction argument, we have that $I_{A_1\cup\dots\cup A_n} \in \mathcal{F}$, $\forall n$

Moreover, $\left(\lim\limits_{n\to\infty} I_{A_1\cup\dots\cup A_n}\right)(x) = I_{\bigcup_{n=1}^{\infty} A_n}(x)$ $\forall x \in X$, so $I_{\bigcup_{n=1}^{\infty} A_n} \in \mathcal{F}$, hence $\bigcup_{n=1}^{\infty} A_n \in \mathcal{M}$.

Thus $\mathcal{M}$ is a σ-algebra of subsets of X. Since $\mathcal{M}$ contains all closed subsets of X, it must also contain all open ones, and so $\mathcal{B}_X \subseteq \mathcal{M}$.

<u>Claim 3:</u> $\mathrm{Bor}_b(X, \mathbb{R}) \subseteq \mathcal{F}$.

proof: Let $f \in \mathrm{Bor}_b(X, \mathbb{R})$. By the result of A4Q4, we can find $(g_n)_{n=1}^{\infty}$ in $\mathrm{Bor}_s(X, \mathbb{R})$ such that $|f(x) - g_n(x)| < \frac{1}{n}$ $\forall x \in X$, $\forall n \in \mathbb{N}$.

Since $\mathrm{Bor}_s(X, \mathbb{R}) = \mathrm{span}\{I_A \mid A \in \mathcal{B}_X\}$ (A4Q3), we have that $\mathrm{Bor}_s(X, \mathbb{R}) \subseteq \mathcal{F}$ by Claim 2, and so $(g_n)_{n=1}^{\infty}$ is in $\mathcal{F}$.

Since $g_n(x) \xrightarrow[n\to\infty]{} f(x)$ $\forall x \in X$, we have that $f \in \mathcal{F}$.

Hence $\mathrm{Bor}_b(X, \mathbb{R}) \subseteq \mathcal{F}$. ▨

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Definition.. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f, g be two functions in $\mathrm{Bor}(X, \mathbb{R})$. If there exists a set $N \in \mathcal{M}$ such that $\mu(N) = 0$ and such that $f(x) = g(x)$ for every $x \in X \setminus N$, then we say that f *and* g *coincide almost everywhere–*μ, and we write "$f = g$ a.e.–μ".

[Note: If $f, g \in \mathrm{Bor}(X, \mathbb{R})$, then the set $\{x \in X \mid f(x) \neq g(x)\}$ is sure to belong to $\mathcal{M}$, because it can be written in the form $(f - g)^{-1}(\mathbb{R} \setminus \{0\})$, where $f - g \in \mathrm{Bor}(X, \mathbb{R})$. It is immediate that the above definition could have been phrased by saying "f and g coincide a.e.–μ if and only $\mu\big(\{x \in X \mid f(x) \neq g(x)\}\big) = 0$."]

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f be a function in $\mathrm{Bor}^+(X, \mathbb{R})$.

(a) Suppose that $f = 0$ a.e.–μ. Prove that $\int f \, d\mu = 0$.

(b) Conversely, suppose that $\int f \, d\mu = 0$. Prove that $f = 0$ a.e.–μ.

Solution.

(a) Let $g \in \mathrm{Bor}_s^+(X, \mathbb{R})$ with $g \leq f$. Then writing g canonically as $g = \sum_{k=1}^n \alpha_k I_{A_k}$ with $\alpha_1, \ldots, \alpha_n \in (0, \infty)$ and $A_1, \ldots, A_n \in \mathcal{B}_X$ with $A_i \cap A_j = \emptyset$ for $i \neq j$, we have that each $A_k \subseteq g^{-1}(0, \infty) \subseteq f^{-1}(0, \infty) = (f - 0)^{-1}(\mathbb{R} \setminus \{0\})$ (since $g \leq f$), whence

$$L_s^+(g) = \sum_{k=1}^n \alpha_k \mu(A_k) \leq \sum_{k=1}^n \alpha_k \mu(f^{-1}(0, \infty)) = \sum_{k=1}^n \alpha_k \cdot 0 = 0.$$

Thus,

$$\int f d\mu = \sup\{L_s^+(g) \mid g \in \mathrm{Bor}_s^+(X, \mathbb{R}), g \leq f\} = \sup\{0 \mid g \in \mathrm{Bor}_s^+(X, \mathbb{R}), g \leq f\} = 0. \quad \blacksquare$$

(b) We prove the contrapositive. Suppose that $f \neq 0$ a.e.–μ, so $0 < \mu((f-0)^{-1}(\mathbb{R} \setminus \{0\})) = \mu(f^{-1}(0, \infty))$. For each $n \in \mathbb{N}$, let $S_n := f^{-1}(\frac{1}{n}, \infty)$. Note that $S_1 \subseteq S_2 \subseteq \cdots \subseteq f^{-1}(0, \infty)$ with $\bigcup_{n=1}^{\infty} S_n = f^{-1}(0, \infty)$, so by continuity along chains, $\lim_{n \to \infty} \mu(S_n) = \mu(f^{-1}(0, \infty))$. Since $\mu(f^{-1}(0, \infty)) > 0$, there is a $N \in \mathbb{N}$ such that $\mu(S_N) > 0$. Then the function $\frac{1}{N} I_{S_N} \in \mathrm{Bor}_s^+(X, \mathbb{R})$ with $\frac{1}{N} I_{S_N} \leq f$ since for any $x \in S_N = f^{-1}(\frac{1}{N}, \infty)$, $\frac{1}{N} I_{S_N}(x) = \frac{1}{N} < f(x)$. However, $L_s^+(\frac{1}{N} I_{S_N}) = \frac{1}{N}\mu(S_N) > 0$, so

$$\int f d\mu = \sup\{L_s^+(g) \mid g \in \mathrm{Bor}_s^+(X, \mathbb{R}), g \leq f\} \geq L_s^+\left(\frac{1}{N} I_{S_N}\right) > 0.$$

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Thus, if $\int f d\mu = 0$, then we must have $f = 0$ a.e.–μ. $\blacksquare$

For Problem 3 we fix a measure space $(X, \mathcal{M}, \mu)$ where μ is a finite positive measure ($\mu(X) < \infty$). We also fix a function $f \in \mathrm{Bor}^+(X, \mathbb{R})$. We assume that f is bounded; that is, there exists $c \geq 0$ such that $0 \leq f(x) \leq c$ for all $x \in X$. The definition of the Lebesgue integral used in Lecture 8 assigns to f a finite integral

$$I_{Leb} := \int f \, d\mu$$

(it is immediate that $I_{Leb} \leq c \cdot \mu(X)$, with c as in the preceding sentence).

On the other hand, the considerations from Lecture 2 also apply to f and give us another possible approach to the integral, namely we can take the number

$$I_{Dar} := \overline{\int} f \, d\mu = \underline{\int} f \, d\mu.$$

Here the upper and lower integral of f are defined as in Lecture 2, and the fact that they are equal to each other was the content of Proposition 2.6 of that lecture.

Your goal for Problem 3 is to show that the two approaches to the integral of f give the same result.

Problem 3. In the framework and notations described above, prove that $I_{Leb} = I_{Dar}$.

Solution.

Consider the a measurable division $\Delta = \{A_1, \ldots, A_n\}$ of X (w.r.t. $\mathcal{M}$). Let $f(\Delta) = \sum_{i=1}^n (\inf_{A_i} f) I_{A_i} \in \mathrm{Bor}_s^+(X, \mathbb{R})$. It is by definition clear that $f \geq f(\Delta)$. Therefore, $I_{Leb} \geq L_s^+(f(\Delta)) = \sum_{i=1}^n (\inf_{A_i} f)\mu(A_i) = L(f, \Delta)$. As this is true for all divisions Δ of X, it follows that $I_{Leb} \geq I_{Dar}$.

On the other hand, let $g(\Delta) = \sum_{i=1}^n (\sup_{A_i} f) I_{A_i} \in \mathrm{Bor}_s^+(X, \mathbb{R})$ (note f and therefore g are bounded). It is by definition clear that $f \leq g(\Delta)$. Hence, by the increasing property of the Lebesgue integral, and the fact that the Lebesgue integral extends the "natural" integral on simple functions, we get that

$$I_{Leb} \leq \int g(\Delta) = L_s^+(g(\Delta)) = \sum_{i=1}^n (\sup_{A_i} f)\mu(A_i) = U(f, \Delta).$$

As this is true for all divisions Δ of X, it follows that $I_{Leb} \leq I_{Dar}$.

The two inequalities yield $I_{Leb} = I_{Dar}$, as desired. $\square$

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