PMath 451/651, Fall Term 2012

Homework Assignment 6 – Solutions

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let f, g be two functions in $\mathcal{L}^1(\mu)$.

(a) Suppose that $f = g$ a.e.–μ (in the sense defined in Problem 2 of Assignment 5). Prove that $\int_A f\, d\mu = \int_A g\, d\mu$ for all $A \in \mathcal{M}$.

(b) Conversely, suppose that $\int_A f\, d\mu = \int_A g\, d\mu$ for all $A \in \mathcal{M}$. Prove that $f = g$ a.e.–μ.

Solution. (a) Let A be a set in $\mathcal{M}$. Consider the function $h := |f-g| \cdot I_A$. This is a Borel function, since it is obtained from f, g and I_A by operations which preserve measurability. It is obvious that h is non-negative. Moreover, let us also observe that h is equal to 0 a.e.–μ; indeed, if we denote $\{x \in X \mid f(x) \neq g(x)\} =: N$, then $\mu(N) = 0$ due to the hypothesis that $f = g$ a.e.–μ, and h vanishes on the complement of N.

We next observe that

$$
\begin{aligned}
\Big| \int_A f\, d\mu - \int_A g\, d\mu \Big| &= \Big| \int f\, I_A\, d\mu - \int g I_A\, d\mu \Big| \\
&= \Big| \int (f-g)\, I_A\, d\mu \Big| \\
&\leq \int |f-g|\, I_A\, d\mu = \int h\, d\mu = 0,
\end{aligned}
$$

where at the last step we use Problem 2(a) from assignment 5.

It follows that $| \int_A f\, d\mu - \int_A g\, d\mu | = 0$, hence that $\int_A f\, d\mu = \int_A g\, d\mu$, as required.

(b) Let us denote $f - g =: u$. Then $u \in \mathcal{L}^1(\mu)$, and for every $A \in \mathcal{M}$ we have that

$$\int_A u\, d\mu = \int_A f\, d\mu - \int_A g\, d\mu = 0. \tag{1}$$

In equation (1) let us consider the special cases when the set $A \in \mathcal{M}$ being considered is either $P := \{x \in X \mid u(x) > 0\}$ or $Q := \{x \in X \mid u(x) < 0\}$. We obtain that

$$\int_P u\, d\mu = \int_Q u\, d\mu = 0,$$

hence that

$$\int u\, I_P\, d\mu = \int u\, I_Q\, d\mu = 0,$$

which implies that

$$\int u \cdot \big(I_P - I_Q\big)\, d\mu = 0. \tag{2}$$

From how P and Q are defined, it is immediate that $u \cdot (I_P - I_Q) = |u|$; so in (2) we have actually obtained that $\int |u|\, d\mu = 0$.

We can now apply Problem 2(b) from homework assignment 5 to the function $|u|$, and this gives us that the set $N := \{x \in X \mid u(x) \neq 0\}$ has $\mu(N) = 0$. But since $u = f - g$, the set N is nothing but $\{x \in X \mid f(x) \neq g(x)\}$, and in this way we obtain that $f = g$ a.e.–μ, as required.

Problem 2 is about Lebesgue-Stieltjes measures on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ of the real line.

If $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ is a Lebesgue-Stieltjes measure and if G is the centered Stieltjes function associated to μ, it is customary (especially in probability textbooks) to use the notation "$\int f(t) d\, G(t)$" instead of "$\int f(t) d\, \mu(t)$", for a function $f \in \mathcal{L}^1(\mu)$. This notation can be further shortened by saying (somewhat abusively) that we have "$d\, \mu(t) = d\, G(t)$".

We will denote by λ the Lebesgue measure on $\mathbb{R}$. The centered Stieltjes function of λ is the identity function, hence the shortened notation from the preceding paragraph comes in this case to "$d\, \lambda(t) = dt$".

With all these shortened notations, Problem 2 can be summarized by the formula "$d\, G(t) = G'(t)\, dt$".

Problem 2. Let $G : \mathbb{R} \to \mathbb{R}$ be a differentiable non-decreasing function such that $G(0) = 0$. We denote $G' =: g$. On the other hand let $\mu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ be the Lebesgue-Stieltjes measure which is uniquely determined by the requirement that the centered Stieltjes function of μ is equal to G.

(a) Prove that g is a non-negative Borel function.

(b) Suppose that g is bounded on every interval $[a, b] \subseteq \mathbb{R}$. Let $\nu : \mathcal{B}_{\mathbb{R}} \to [0, \infty]$ be the positive measure defined (in the sense of Lecture 11) by the formula "$d\, \nu(t) = g(t) d\, \lambda(t)$", where λ is the Lebesgue measure on $\mathbb{R}$. That is, we put $\nu(A) := \int_A g\, d\lambda$, $\forall\, A \in \mathcal{B}_{\mathbb{R}}$. Prove that $\nu = \mu$.

Solution. (a) For every $n \in \mathbb{N}$ let us define $g_n : \mathbb{R} \to \mathbb{R}$ by

$$g_n(t) = \frac{G(t + \frac{1}{n}) - G(t)}{\frac{1}{n}}, \quad t \in \mathbb{R}.$$

The function g_n is continuous on $\mathbb{R}$ (indeed, it is immediate to verify that g_n respects the convergence of sequences in $\mathbb{R}$). In particular it follows that g_n is a Borel function.

Now, from the definition of the derivative it is clear that for every fixed $t \in \mathbb{R}$ we have $\lim_{n \to \infty} g_n(t) = G'(t) = g(t)$. Thus the function g is the pointwise limit of the g_n's. Since a poinwise limit of a sequence of Borel functions is still a Borel function, we conclude that g is a Borel function. Finally, we note that g is non-negative, due to the fact that G is non-decreasing.

(b) We first prove that ν is a Lebesgue-Stieltjes measure. To this end, it suffices to fix some numbers $a < b$ in $\mathbb{R}$ and to prove that $\nu(\, [a, b]\,) < \infty$. And indeed, if $\gamma \geq 0$ is picked such that $g(t) \leq \gamma$ for all $t \in [a, b]$, then we have

$$\nu(\, [a, b]\,) = \int_{[a,b]} g(t) d\, \lambda(t) \leq \int_{[a,b]} \gamma d\, \lambda(t) = \gamma \cdot (b - a) < \infty.$$

We know that two Lebesgue-Stieltjes measures coincide when they have the same centered Stieltjes function. Thus in order to prove that $\mu = \nu$, it will suffice to prove that μ and ν have the same centered Stieltjes function. The centered Stieltjes function of μ

is (by definition) equal to G, hence what we have to show is that the centered Stieltjes function of ν is G. This in turn, amounts to proving that for every $a < b$ in $\mathbb{R}$ we have $\nu(\,(a,b]\,) = G(b) - G(a)$. Keeping in mind how ν is defined, we are thus left to verify that

$$(*) \qquad \int_{(a,b]} g(t)\, d\lambda(t) = G(b) - G(a), \quad \forall\, a < b \text{ in } \mathbb{R}.$$

For the remaining part of the solution we fix $a < b$ in $\mathbb{R}$ for which we will verify that $(*)$ holds. We will use an anti-derivative for G. More precisely, let $H : \mathbb{R} \to \mathbb{R}$ be the unique differentiable function on $\mathbb{R}$ which has the properties that $H' = G$ and $H(0) = 0$. By the fundamental theorem of calculus for continuous functions, for every $p < q$ in $\mathbb{R}$ we have

$$H(q) - H(p) = \int_p^q G(t)\, dt \quad \text{(Riemann integral).}$$

We will also consider the functions $(g_n)_{n=1}^{\infty}$ that were used in the solution to part (a). We will find an explicit formula for the integrals of these functions on $(a,b]$. Since g_n is continuous, its Lebesgue integral on $(a,b]$ coincides with its Riemann integral from a to b:

$$\int_{(a,b]} g_n\, d\lambda = \int_a^b g_n(t)\, dt.$$

The formula that defined g_n in part (a) can be re-written in the form

$$g_n = n(G_n - G),$$

where $G_n : \mathbb{R} \to \mathbb{R}$ is defined by

$$G_n(t) = G\big(\, t + \frac{1}{n}\,\big), \quad t \in \mathbb{R}.$$

It is immediate that for any $p < q$ in $\mathbb{R}$ we have

$$\int_p^q G_n(t)\, dt = \int_{p+1/n}^{q+1/n} G(s)\, ds = H(q + 1/n) - H(p + 1/n).$$

So then we have

$$\begin{aligned} \int_a^b g_n(t) &= n \cdot \int_a^b G_n(t)\, dt - n \cdot \int_a^b G(t)\, dt \\ &= n\big(H(b + 1/n) - H(a + 1/n)\big) - n\big(H(b) - H(a)\big) \end{aligned}$$

By writing the latter expression as

$$\frac{H(b + \frac{1}{n}) - H(b)}{\frac{1}{n}} - \frac{H(a + \frac{1}{n}) - H(a)}{\frac{1}{n}},$$

we find that

$$\lim_{n \to \infty} \int_a^b g_n(t)\, dt = H'(b) - H'(a) = G(b) - G(a).$$

Now let us recall (from the solution to part (a)) that the functions g_n converge pointwise to g. If we can prove that Lebesgue dominated convergence theorem applies to this situation, then it will follow that

$$\int_{(a,b]} g(t)\, d\lambda(t) = \lim_{n\to\infty} \int_{(a,b]} g_n(t)\, d\lambda(t) = \lim_{n\to\infty} \int_a^b g_n(t)\, dt = G(b) - G(a),$$

which is the required formula $(*)$. Thus in order to finish the solution, we are only left to find a dominating function for the g_n's, on the interval $(a,b]$. Since $\lambda(\,(a,b]\,) = b - a < \infty$, we can go for a constant dominating function; that is, it suffices to find a constant $\gamma \geq 0$ such that $g_n(t) \leq \gamma$ for all $n \in \mathbb{N}$ and $t \in (a,b]$.

Finally, let us use the hypothesis that g is bounded on bounded intervals and let us pick γ such that $g(t) \leq \gamma$ for every $t \in [a, b+1]$. We claim that this γ does the dominating job that we need. Indeed, given $n \in \mathbb{N}$ and $t \in (a,b]$, there exists (by the mean value theorem applied to G on the interval $[t, t+\,1/n]$) an intermediate value $s \in \big(t, t+\frac{1}{n}\big) \subseteq [a, b+1]$ such that $g_n(t) = G'(s)$, and thus we have $g_n(t) = g(s) \leq \gamma$.

In Problem 3 we fix a measurable space $(X,\mathcal{M})$ and we consider the space of finite signed measures $\mathrm{Meas}^{\pm}(X,\mathcal{M})$ that was introduced in Lecture 11. Recall that $\mathrm{Meas}^{\pm}(X,\mathcal{M})$ is a vector space over $\mathbb{R}$, with operations defined in the natural way: $(\alpha_1\nu_1+\alpha_2\nu_2)(A) := \alpha_1\nu_1(A)+\alpha_2\nu_2(A)$, for every $\nu_1,\nu_2 \in \mathrm{Meas}^{\pm}(X,\mathcal{M})$, $\alpha_1,\alpha_2 \in \mathbb{R}$, $A \in \mathcal{M}$. It will be convenient to also use the partial order on $\mathrm{Meas}^{\pm}(X,\mathcal{M})$ defined as follows: for $\nu_1,\nu_2 \in \mathrm{Meas}^{\pm}(X,\mathcal{M})$ we write "$\nu_1 \leq \nu_2$" to mean that $\nu_1(A) \leq \nu_2(A)$ for every $A \in \mathcal{M}$. Equivalently, one can say that

$$\big(\nu_1 \leq \nu_2\big) \Leftrightarrow \big(\nu_2 - \nu_1 \in \mathrm{Meas}^{+}(X,\mathcal{M})\big),$$

where $\mathrm{Meas}^{+}(X,\mathcal{M})$ is the subset of $\mathrm{Meas}^{\pm}(X,\mathcal{M})$ consisting of positive finite measures.

Problem 3. (a) Let ν be in $\mathrm{Meas}^{\pm}(X,\mathcal{M})$, and consider the total variation measure $|\nu| \in \mathrm{Meas}^{+}(X,\mathcal{M})$. Prove that for every $A \in \mathcal{M}$ one has

$$|\nu|(A) = \sup\big\{\, |\nu(B)| + |\nu(C)| \;\mid\; B,C \in \mathcal{M},\ B,C \subseteq A,\ B \cap C = \emptyset \big\}.$$

(b) Let ν be a measure in $\mathrm{Meas}^{\pm}(X,\mathcal{M})$. Suppose σ is a measure in $\mathrm{Meas}^{+}(X,\mathcal{M})$ with the property that $\sigma(A) \geq |\nu(A)|, \ \ \forall\, A \in \mathcal{M}$. Prove that $\sigma \geq |\nu|$.

(c) Let ν be a measure in $\mathrm{Meas}^{\pm}(X,\mathcal{M})$ and let α be a real number. Prove that $|\,\alpha\nu\,| = |\alpha| \cdot |\nu|$.

(d) Let ν_1,ν_2 be two measures in $\mathrm{Meas}^{\pm}(X,\mathcal{M})$. Prove that $|\nu_1+\nu_2| \leq |\nu_1| + |\nu_2|$.

(e) For every $\nu \in \mathrm{Meas}^{\pm}(X,\mathcal{M})$, we denote $||\nu|| := |\nu|(X)$. Prove that $\nu \mapsto ||\nu||$ is a norm on the vector space $\mathrm{Meas}^{\pm}(X,\mathcal{M})$.

Solution. (a) We divide the proof into three claims.

Claim 1. For every $B \in \mathcal{M}$, one has that $|\,\nu(B)\,| \leq |\nu|(B)$.
Verification of Claim 1. We distinguish two cases.
Case 1: $\nu(B) \geq 0$. In this case we write

$$\begin{aligned} |\,\nu(B)\,| &= \nu(B) \\ &\leq V^{+}(B) \quad \text{(by definition of } V^{+}\text{)} \\ &\leq |\nu|(B) \quad \text{(since } V^{+} \leq |\nu|\text{)}. \end{aligned}$$

Case 2: $\nu(B) < 0$. In this case we write

$$\begin{aligned} |\,\nu(B)\,| &= -\nu(B) \\ &\leq V^{-}(B) \quad \text{(by definition of } V^{-}\text{)} \\ &\leq |\nu|(B) \quad \text{(since } V^{-} \leq |\nu|\text{)}. \end{aligned}$$

This completes the verification of Claim 1.

Claim 2. Let A be a set in $\mathcal{M}$. Suppose that $B,C \in \mathcal{M}$ are such that $B,C \subseteq A$ and $B \cap C = \emptyset$. Then $|\,\nu(B)\,| + |\,\nu(C)\,| \leq |\nu|(A)$.

Verification of Claim 2. We write

$$\begin{aligned} |\,\nu(B)\,| + |\,\nu(C)\,| &\le |\nu|(B) + |\nu|(C) \quad \text{(by Claim 1)} \\ &= |\nu|(B \cup C) \quad \text{(because } |\nu| \text{ is additive)} \\ &\le |\nu|(A), \end{aligned}$$

where the last inequality holds because $B \cup C \subseteq A$ and because $|\nu|$ is an increasing set-function.

Claim 3. Let A be a set in $\mathcal{M}$. One can find sets $B, C \in \mathcal{M}$ such that $B, C \subseteq A$, such that $B \cap C = \emptyset$, and such that $|\,\nu(B)\,| + |\,\nu(C)\,| = |\nu|(A)$.

Verification of Claim 3. Let (Y^+, Y^-) be a Hahn decomposition for ν, and let us put

$$B := A \cap Y^+, \quad C := A \cap Y^-.$$

Then $B, C \in \mathcal{M}$ and we have $B \cup C = A$, $B \cap C = \emptyset$). Moreover, from the properties of a Hahn decomposition we infer that

$$B \subseteq Y^+ \Rightarrow \Big(\nu(B) = V^+(B) \text{ and } V^-(B) = 0 \Big), \tag{3}$$

while on the other hand

$$C \subseteq Y^- \Rightarrow \Big(\nu(C) = -V^-(C) \text{ and } V^+(C) = 0 \Big). \tag{4}$$

From (3) it follows that

$$|\,\nu(B)\,| = V^+(B) = V^+(B) + V^-(B) = |\nu|(B);$$

likewise, from (3) it follows that

$$|\,\nu(C)\,| = V^-(C) = V^+(C) + V^-(C) = |\nu|(C).$$

We thus conclude that

$$\begin{aligned} |\,\nu(B)\,| + |\,\nu(C)\,| &= |\nu|(B) + |\nu|(C) \quad \text{(by the above)} \\ &= |\nu|(B \cup C) \quad \text{(because } |\nu| \text{ is additive)} \\ &= |\nu|(A) \quad \text{(because } B \cup C = A\text{)}. \end{aligned}$$

So B and C have all the properties required in Claim 3.

The statement of part (a) of the problem follows immediately from the Claims 2 and 3 proved above.

(b) Fix a set $A \in \mathcal{M}$, for which we will verify the inequality $\sigma(A) \ge |\nu|(A)$. In view of how $|\nu|(A)$ is described in part (a) of the problem, it suffices to verify that

$$\sigma(A) \ge |\,\nu(B)\,| + |\,\nu(C)\,|$$

whenever $B, C \in \mathcal{M}$ are such that $B, C \subseteq A$ and $B \cap C = \emptyset$. And indeed, for any such B and C we have

$$\begin{aligned} |\,\nu(B)\,| + |\,\nu(C)\,| &\leq \sigma(B) + \sigma(C) \quad \text{(by hypothesis given on } \sigma) \\ &= \sigma(B \cup C) \quad \text{(because } \sigma \text{ is additive)} \\ &\leq \sigma(A), \end{aligned}$$

where the last inequality holds because $B \cup C \subseteq A$ and σ is an increasing set-function.

(c) The case when $\alpha = 0$ is immediate, because in this case both $|\,\alpha\nu\,|$ and $|\alpha| \cdot |\nu|$ are the zero measure. So we will assume that $\alpha \neq 0$. We will let (as usual) V^+ and V^- denote the positive and respectively negative variations of ν. On the other hand, we will let W^+ and W^- denote the positive and respectively negative variations of $\alpha\,\nu$. We have two cases:

Case 1: $\alpha > 0$. Directly from the definitions of positive/negative variations, it is immediate that in this case we have

$$W^+ = \alpha V^+ \ \text{ and } W^- = \alpha V^-.$$

Therefore

$$|\,\alpha\nu\,| = W^+ + W^- = \alpha V^+ + \alpha V^- = \alpha|\nu|,$$

as required.

Case 2: $\alpha < 0$. Still directly from the definitions of positive/negative variations, it is immediate that in this case we have

$$W^+ = |\alpha|\,V^- \ \text{ and } W^- = |\alpha|\,V^+.$$

Therefore

$$|\,\alpha\nu\,| = W^+ + W^- = |\alpha|\,V^- + |\alpha|\,V^+ = |\alpha| \cdot |\nu|,$$

as required.

(d) We denote $|\nu_1| + |\nu_2| =: \sigma \in \mathrm{Meas}^+(X, \mathcal{M})$. We have to prove that $|\nu_1 + \nu_2| \leq \sigma$. In view of part (b) of the question, it will suffice to verify that

$$|(\nu_1 + \nu_2)(A)| \leq \sigma(A), \quad \forall\, A \in \mathcal{M}.$$

So let us fix an $A \in \mathcal{M}$. We write:

$$\begin{aligned} |(\nu_1 + \nu_2)(A)| &= |\nu_1(A) + \nu_2(A)| \\ &\leq |\nu_1(A)| + |\nu_2(A)| \\ &\leq |\nu_1|(A) + |\nu_2|(A) \\ &= \sigma(A), \end{aligned}$$

where at the second inequality sign we used the property of the total variation measure which was proved at the beginning of the solution to part (a) (cf. Claim 1 in the argument shown there). This concludes the verification needed for part (d).

(e) Positivity: For every $\nu \in \mathrm{Meas}^{\pm}(X,\mathcal{M})$ we have $||\nu|| := |\nu|(X) \geq 0$. If $||\nu|| = 0$, then for an arbitrary $B \in \mathcal{M}$ we can write

$$0 \leq |\nu(B)| \leq |\nu|(B) \leq |\nu|(X) = ||\nu|| = 0$$

(with the second inequality taken from Claim 1 of the solution to part (a)); this implies that $\nu(B) = 0$ for all $B \in \mathcal{M}$, hence that ν is the zero-vector of the space $\mathrm{Meas}^{\pm}(X,\mathcal{M})$.

Homogeneity: For $\alpha \in \mathbb{R}$ and $\nu \in \mathrm{Meas}^{\pm}(X,\mathcal{M})$ we have

$$\begin{aligned} ||\alpha\,\nu|| &= |\alpha\,\nu|(X) \\ &= |\alpha| \cdot |\nu|(X) \quad \text{(by (c))} \\ &= |\alpha| \cdot ||\nu||. \end{aligned}$$

Triangle inequality: for $\nu_1, \nu_2 \in \mathrm{Meas}^{\pm}(X,\mathcal{M})$ we have

$$\begin{aligned} ||\nu_1 + \nu_2|| &= |\nu_1 + \nu_2|(X) \\ &\leq (|\nu_1| + |\nu_2|)(X) \quad \text{(by (d))} \\ &= |\nu_1|(X) + |\nu_2|(X) \\ &= ||\nu_1|| + ||\nu_2||. \end{aligned}$$