PMath 451/651, Fall Term 2012

Homework Assignment 7 – Solutions

Problem 1 asks you to verify a general statement about Borel functions which was accepted (without proof) in class, in the lecture on the Radon-Nikodym theorem.

Problem 1. Let $(X, \mathcal{M})$ be a measurable space. Suppose we are given a family $\{B_t \mid t \in \mathbb{Q} \cap [0,1]\}$ of sets from $\mathcal{M}$ such that $B_1 = X$ and such that

$$\left(\begin{array}{c} s,t \in \mathbb{Q} \cap [0,1] \\ \text{and } s < t \end{array} \right) \Rightarrow \ B_s \subseteq B_t.$$

Prove that there exists a function $g \in \mathrm{Bor}(X, \mathbb{R})$, with $0 \leq g(x) \leq 1$ for every $x \in X$, and such that for every $t \in \mathbb{Q} \cap [0,1]$ we have the implications:

$$\begin{cases} x \in B_t & \Rightarrow \quad g(x) \leq t \\ x \in X \setminus B_t & \Rightarrow \quad g(x) \geq t. \end{cases}$$

Solution. For every $x \in X$, we define

$$g(x) := \inf\{t \in \mathbb{Q} \cap [0,1] \mid x \in B_t\}. \tag{1}$$

This definition makes sense because $\{t \in \mathbb{Q} \cap [0,1] \mid x \in B_t\}$ is a nonempty subset of $[0,1]$ (we know for sure it is nonempty, since it contains the number 1); so the infimum of the set is well-defined, and is some number in $[0,1]$.

Equation (1) defines a function $g : X \to \mathbb{R}$. It is obvious that $0 \leq g(x) \leq 1$ for every $x \in X$.

Claim 1. g is a Borel function.

Verification of Claim 1. For every $t \in \mathbb{Q} \cap [0,1]$ let us consider the function

$$f_t := tI_{B_t} + I_{X \setminus B_t}.$$

In other words, f_t is defined such that $f_t(x) = t$ for every $x \in B_t$ and such that $f_t(x) = 1$ for every $x \in X \setminus B_t$. The definition of the function g in (1) can be rephrased in the form

$$g(t) = \inf\{f_t(x) \mid t \in \mathbb{Q} \cap [0,1]\}.$$

Now, every f_t is a Borel function (because it is obtained by algebraic operations from the indicator functions of B_t and $X \setminus B_t$). Since it was proved in class that the infimum of a countable collection of Borel functions is still a Borel function, we conclude that g is a Borel function as well.

Claim 2. g satisfies the implications required in the statement of the problem.

Verification of Claim 2. Fix a number $t_o \in \mathbb{Q} \cap [0,1]$ and an element $x \in X$. We have two cases to consider.

Case 1. $x \in B_{t_o}$. In this case the infimum defining $g(x)$ in (1) includes the number t_o, and it follows that $g(x) \leq t_o$.

Case 2. $x \notin B_{t_o}$. In this case we have to verify that $g(x) \geq t_o$. Assume by contradiction that $g(x) < t_o$. From the definition of $g(x)$ as an infimum it then follows that there exists $t < t_o$ in $\mathbb{Q} \cap [0, 1]$ such that $x \in B_t$. For this t we must have $B_t \subseteq B_{t_o}$ (because $t < t_o$ and by the hypothesis of how the sets B_t are included inside each other). So then we get $x \in B_t \subseteq B_{t_o}$, in contradiction to the fact that $x \notin B_{t_o}$. Thus the assumption $g(x) < t_o$ leads to contradiction, and it follows that $g(x) \geq t_o$, as we had to prove.

In Problem 2 we consider the framework used in class for the Lebesgue decomposition theorem. The problem asks you to prove the uniqueness of the decomposition. (Note: the proof of uniqueness doesn't require μ to be σ-finite. But you are welcome to add the finiteness or σ-finiteness of μ to the hypotheses of the problem, if you find that to be useful.)

Problem 2. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let ν be a measure in $\mathrm{Meas}^+(X, \mathcal{M})$. Suppose that the measures $\nu_1, \nu_2, \sigma_1, \sigma_2 \in \mathrm{Meas}^+(X, \mathcal{M})$ satisfy the following conditions:

(i) $\nu_1 + \nu_2 = \nu = \sigma_1 + \sigma_2$;

(ii) $\nu_1 \ll \mu$ and $\sigma_1 \ll \mu$;

(iii) $\nu_2 \perp \mu$ and $\sigma_2 \perp \mu$.

Prove that $\nu_1 = \sigma_1$ and $\nu_2 = \sigma_2$.

Solution. We divide the argument into three claims.

Claim 1. There exists a set $N \in \mathcal{M}$ such that $\mu(N) = 0$ and such that both ν_2 and σ_2 are concentrated on N.

Verification of Claim 1. From the hypothesis that $\nu_2 \perp \mu$ we infer the existence of $N' \in \mathcal{M}$ such that ν_2 is concentrated on N' and μ is concentrated on $X \setminus N'$. The latter condition simply means that $\mu(N') = 0$.

Likewise, from the hypothesis that $\sigma_2 \perp \mu$ we infer the existence of $N'' \in \mathcal{M}$ with $\mu(N'') = 0$ and such that σ_2 is concentrated on N'.

Let us put $N = N' \cup N'' \in \mathcal{M}$. Then $\mu(N) = 0$ (because $0 \leq \mu(N) \leq \mu(N') + \mu(N'') = 0$). On the other hand we have that ν_2 is concentrated on N (because ν_2 is concentrated on $N' \subseteq N$) and that σ_2 is concentrated on N (because σ_2 is concentrated on $N'' \subseteq N$). So N has all the properties required in Claim 1.

Claim 2. Let the set N be as in Claim 1. Then for every $A \in \mathcal{M}$ we have that

$$\nu_1(A) = \nu(A \setminus N) = \sigma_1(A).$$

Verification of Claim 2. Let A be a set in $\mathcal{M}$. We have $A = (A \cap N) \cup (A \setminus N)$, disjoint union, hence

$$\nu_1(A) = \nu_1(A \cap N) + \nu_1(A \setminus N). \tag{2}$$

We know that $\mu(N) = 0$, which implies that $\mu(A \cap N) = 0$ as well (since $0 \leq \mu(A \cap N) \leq \mu(N) = 0$). But $\nu_1 \ll \mu$; so it follows that $\nu_1(A \cap N) = 0$, and Equation (2) reduces to

$$\nu_1(A) = \nu_1(A \setminus N). \tag{3}$$

We next use the fact that ν_2 is concentrated on N, which means by definition that $\nu_2(X \setminus N) = 0$. Since $A \setminus N$ is a subset of $X \setminus N$, it follows that $\nu_2(A \setminus N) = 0$ as well. But then in (3) we can continue as follows:

$$\begin{aligned} \nu_1(A) &= \nu_1(A \setminus N) \\ &= \nu_1(A \setminus N) + \nu_2(A \setminus N) \\ &= \nu(A \setminus N) \quad (\text{since } \nu_1 + \nu_2 = \nu). \end{aligned}$$

This proves the first of the two equalities stated in Claim 2. The verification of the equality $\sigma_1(A) = \nu(A \setminus N)$ is done in exactly the the same way, by replacing ν_1 and ν_2 with σ_1 and σ_2 throughout the argument.

Claim 3. $\nu_1 = \sigma_1$ and $\nu_2 = \sigma_2$.

Verification of Claim 3. The equality $\nu_1 = \sigma_1$ is given by Claim 2. The equality $\nu_2 = \sigma_2$ then also follows, since $\nu_2 = \nu - \nu_1 = \nu - \sigma_1 = \sigma_2$.

The verification of Claim 3 concludes the solution to this problem.

In Problem 3 we use the notation $\mathcal{B}_n$ for the Borel σ-algebra of $(\mathbb{R}^n, d_n)$, where d_n is the Euclidean distance on $\mathbb{R}^n$ (for some positive integer n).

Also, for $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ we will denote, as is usual,

$$A \times B := \{(a, b) \mid a \in A,\ b \in B\} \subseteq \mathbb{R}^{m+n}.$$

Problem 3. Let m, n be two positive integers.

(a) Prove that if $A \in \mathcal{B}_m$ and $B \in \mathcal{B}_n$, then the Cartesian product $A \times B$ belongs to the Borel σ-algebra $\mathcal{B}_{m+n}$.

(b) Consider the collection $\mathcal{P}$ of subsets of $\mathbb{R}^{m+n}$ defined as follows:

$$\mathcal{P} = \{A \times B \mid A \in \mathcal{B}_m \text{ and } B \in \mathcal{B}_n\}.$$

Prove that the σ-algebra generated by $\mathcal{P}$ is equal to $\mathcal{B}_{m+n}$.

Solution. (a) Let us denote by $\mathcal{Q}_1$ the collection of open intervals of $\mathbb{R}$ defined as follows:

$$\mathcal{Q}_1 = \{(a, b) \mid a < b \text{ in } \mathbb{Q}\} \cup \{\emptyset, \mathbb{R}\}.$$

Moreover, for every $k \in \mathbb{N}$ let us denote

$$\mathcal{Q}_k = \{I_1 \times I_2 \times \cdots \times I_k \mid I_1, \ldots, I_k \in \mathcal{Q}_1\}.$$

Then (as known from preceding analysis courses) for every $k \in \mathbb{N}$ we have that $\mathcal{Q}_k$ is a countable basis of open sets for $(\mathbb{R}^k, d_k)$. In other words, every open subset of $\mathbb{R}^k$ can be written as a (necessarily countable) union of sets from $\mathcal{Q}_k$. An immediate consequence of this fact is that the σ-algebra generated by $\mathcal{Q}_k$ is equal to $\mathcal{B}_k$.

Now let us consider the positive integers m, n that are given in this problem. We will arrive to the statement of part (a) of the problem in two stages, presented in the following two claims.

Claim 1. If $Q \in \mathcal{Q}_m$ and $B \in \mathcal{B}_n$, then $Q \times B \in \mathcal{B}_{m+n}$.

Verification of Claim 1. Fix $Q \in \mathcal{Q}_m$, and consider the collection of sets

$$\mathcal{S} = \{B \in \mathcal{B}_n \mid Q \times B \in \mathcal{B}_{m+n}\}.$$

We have to verify that $\mathcal{S} = \mathcal{B}_n$. Let us observe that $\mathcal{S} \supseteq \mathcal{Q}_n$; indeed, it is clear that if $B \in \mathcal{Q}_n$, then $Q \times B \in \mathcal{Q}_{m+n} \subseteq \mathcal{B}_{m+n}$.

In order to complete the verification of Claim 1, we will prove that $\mathcal{S}$ is a σ-algebra. Once this is done, it will follow that $\mathcal{S}$ contains the σ-algebra generated by $\mathcal{Q}_n$, which is $\mathcal{B}_n$, and the equality $\mathcal{S} = \mathcal{B}_n$ will follow.

So let us verify that $\mathcal{S}$ satisfies the three axioms used to define a σ-algebra.

(AS1) $\mathbb{R}^n \in \mathcal{S}$ because $Q \times \mathbb{R}^n \in \mathcal{Q}_{m+n} \subseteq \mathcal{B}_{m+n}$.

(AS2) Suppose that $B \in \mathcal{S}$, and consider the complement $\mathbb{R}^n \setminus B$. It is immediate that

$$Q \times (\mathbb{R}^n \setminus B) = (Q \times \mathbb{R}^n) \setminus (Q \times B).$$

The sets $Q \times \mathbb{R}^n$ and $Q \times B$ belong to $\mathcal{B}_{m+n}$ (the first one from (AS1), and the second one because of the fact that $B \in \mathcal{S}$). The set-difference $(Q \times \mathbb{R}^n) \setminus (Q \times B)$ must therefore also belong to $\mathcal{B}_{m+n}$, and it follows that $\mathbb{R}^n \setminus B \in \mathcal{S}$.

(σ-AS3). Suppose that $(B_k)_{k=1}^{\infty}$ are from $\mathcal{S}$, and consider the union $B = \cup_{k=1}^{\infty} B_k$. It is immediate that

$$Q \times (\cup_{k=1}^{\infty} B_k) = \cup_{k=1}^{\infty} (Q \times B_k).$$

Each of the sets $Q \times B_k$ belongs to $\mathcal{B}_{m+n}$ (because $B_k \in \mathcal{S}$) and $\mathcal{B}_{m+n}$ is closed under countable unions; hence $\cup_{k=1}^{\infty}(Q \times B_k)$ is in $\mathcal{B}_{m+n}$, which implies that $\cup_{k=1}^{\infty} B_k \in \mathcal{S}$. (This ends the verification of Claim 1.)

Claim 2. If $A \in \mathcal{B}_m$ and $B \in \mathcal{B}_n$, then $A \times B \in \mathcal{B}_{m+n}$.

Verification of Claim 2. Fix $B \in \mathcal{B}_n$, and consider the collection of sets

$$\mathcal{T} = \{A \in \mathcal{B}_m \mid A \times B \in \mathcal{B}_{m+n}\}.$$

We have to verify that $\mathcal{T} = \mathcal{B}_m$.

The collection of sets $\mathcal{T}$ is a a σ-algebra. The proof of this fact is very similar to the proof shown in the verification of Claim 1 (for the collection of sets $\mathcal{S}$ that had appeared there), where now the needed Boolean operations such as complement or union are performed on the first component of the Cartesian products.

On the other hand Claim 1 gives us that $\mathcal{T} \supseteq \mathcal{Q}_m$. So then $\mathcal{T}$ must contain the σ-algebra generated by $\mathcal{Q}_m$, which is $\mathcal{B}_m$. This forces the equality $\mathcal{T} = \mathcal{Q}_m$, and ends the verification of Claim 2 (also ends the solution of part (a) of the problem).

(b) Let $\mathcal{U}$ denote the σ-algebra generated by $\mathcal{P}$. We have to prove that $\mathcal{U} = \mathcal{B}_{m+n}$.

"$\subseteq$" $\mathcal{B}_{m+n}$ is a σ-algebra, and $\mathcal{B}_{m+n}$ contains $\mathcal{P}$ by part (a) of the problem. Hence $\mathcal{B}_{m+n}$ must contain the σ-algebra generated by $\mathcal{P}$, which is $\mathcal{U}$.

"$\supseteq$" By specializing $A \in \mathcal{Q}_m$ and $B \in \mathcal{Q}_n$ in the definition of $\mathcal{P}$, we see that $\mathcal{P} \supseteq \mathcal{Q}_{m+n}$. Hence $\mathcal{U} \supseteq \mathcal{Q}_{m+n}$. Since $\mathcal{U}$ is a σ-algebra, it follows that $\mathcal{U}$ must contain the σ-algebra generated by $\mathcal{Q}_{m+n}$, which is $\mathcal{B}_{m+n}$.